Method of Moments for Estimation

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In statistics, the method of moments is a method of estimation of population parameters such as mean, variance, median, etc. (which need not be moments), by equating sample moments with unobservable population moments and then solving those equations for the quantities to be estimated.

Suppose we have an i.i.d. sample from a distribution with pdf $f(x;\theta)$ that depends on a vector of parameters $\theta = (\theta_1, \ldots, \theta_p)$. The expectation

$$EX^k = \int x^k f(x;\theta)dx$$

is called the kth moment of the distribution or of the population. The kth sample moment is defined as $(1/n)\sum_{i=1}^n X_i^k$, which is an unbiased estimator of EX^k. We solve the p equations

$$EX^k = \frac{1}{n}\sum_{i=1}^n X_i^k, k = 1, 2, \ldots, n.$$

The solution, denoted by $\hat{\theta}$, is called a moments estimator.

Example. Suppose $X_1, \ldots, X_n$ are independent identically distributed random variables with a gamma distribution with probability density function

$$\frac{x^{\alpha-1}e^{-x/\beta}}{\beta^\alpha\,\Gamma(\alpha)}$$

for $x > 0$, and 0 for $x < 0$.

The first moment, i.e., the expected value, of a random variable with this probability distribution is $E(X_1) = \alpha\beta$ and the second moment, i.e., the expected value of its square, is $E(X_1^2) = \beta^2\alpha(\alpha+1)$. These are the population moments.

The first and second sample moments are, respectively,

$$m_1 = \frac{X_1 + \cdots + X_n}{n}$$

and

$$m_2 = \frac{X_1^2 + \cdots + X_n^2}{n}.$$

Equating the population moments with the sample moments, we get

$$\begin{aligned} \alpha\beta &= m_1 \\ \beta^2\alpha(\alpha+1) &= m_2. \end{aligned}$$

Solving these two equations for a and , we get $\alpha = \frac{m_1^2}{m_2 - m_1^2}$ and $\beta = \frac{m_2 - m_1^2}{m_1}$.

Moments estimators in general are inferior to the maximum likelihood estimators. However, in some cases, as in the above example of the gamma distribution, the likelihood equations may be intractable without computers, whereas the method-of-moments estimators can be quickly and easily calculated by hand as shown above.

Estimates by the method of moments may be used as the first approximation to the solutions of the likelihood equations, and successive improved approximations may then be found by the Newton-Raphson method.

In some cases, infrequent with large samples but not so infrequent with small samples, the estimates given by the method of moments are outside of the parameter space; it does not make sense to rely on them then.

One advantage of the moments estimators to the maximum likelihood estimators is that the method of moments requires the first few moments is known (i.e., how they depend on the parameters) while the maximum likelihood method requires the exact parametric expression of the full distribution to be known. So the method of moments requires less assumptions on the populations.