1. Given any variables $X_1, \cdots, X_n$, the best linear prediction for a random variable X is the linear combination $\sum_{i=1}^n a_i X_i$ that minimizes the mean-squared error (assuming the means of all variables are 0),

$$
\begin{aligned}
&E(X - \sum_{i=1}^n a_i X_i)^2 \\
&= Var(X) - 2\sum_{i=1}^n a_i Cov(X, X_i) \\
&+ \sum_{i=1}^n \sum_{j=1}^n a_i a_j Cov(X_i, X_j)
\end{aligned}
$$

Show that the partial derivative of the MSE with respect to a_i is

$$
-2C(X, X_i) + 2\sum_{j=1}^n Cov(X_i, X_j) a_j. \tag{1}
$$

These n equations can be written in matrix notation

$$
R\boldsymbol{a} = \boldsymbol{b} \tag{2}
$$

where R is an $n \times n$ matrix whose (i, j)th element is $Cov(X_i, X_j)$, i.e.,

$$
\begin{aligned}
&R = (Cov(X_i, X_j)), \text{and} \\
&\boldsymbol{b} = (Cov(X, X_1), \ldots, Cov(X, X_n))'. \\
&\boldsymbol{a} = (a_1, \ldots, a_n)'.
\end{aligned}
$$

2. Let us apply the result to a stationary sequence Y_t with mean 0. Write the best linear predictor for Y_{n+1} given $Y_n, \cdots, Y_1$ as

$$
\hat{Y}_{n+1} = \sum_{i=1}^n \phi_{n,i} Y_{n+1-i}.
$$

Write

$$
\begin{aligned}
&\boldsymbol{\phi}_n = (\phi_{n,1}, \cdots, \phi_{n,n})' \text{ note this is a vector} \\
&R_n = (\gamma(i-j))_{n \times n} \\
&k_n = (\gamma(1), \cdots, \gamma(n))'.
\end{aligned}
$$

By (2),

$$
\boldsymbol{\phi}_n = R_n^{-1} k_n. \tag{3}
$$

You can use this to obtain the partial autocorrelation function (PACF) $\phi_{n,n}$, which is the last element of ϕ_n.

3. The Durbin-Levinson Algorithm. There is an iterative way to get the PACF. It goes at follows.

$$\phi_{11} = \gamma(1)/\gamma(0); \upsilon_1 = \gamma(0)(1 - \phi_{11}^2).$$

For $n \geq 1$

$$\phi_{n+1,n+1} = (\gamma(n+1) - \sum_{j=1}^{n} \phi_{n,j}\gamma(n+1-j))/\upsilon_n,$$

$$\phi_{n+1,j} = \phi_{n,j} - \phi_{n+1,n+1}\phi_{n,n-j+1},\ j = 1, 2, \ldots, n,$$

or

$$\begin{pmatrix} \phi_{n+1,1} \\ \vdots \\ \phi_{n+1,n} \end{pmatrix} = \begin{pmatrix} \phi_{n,1} \\ \vdots \\ \phi_{n,n} \end{pmatrix} - \phi_{n+1,n+1} \begin{pmatrix} \phi_{n,n} \\ \vdots \\ \phi_{n,1} \end{pmatrix}, \text{ and}$$

$$\upsilon_{n+1} = \upsilon_n(1 - \phi_{n+1,n+1}^2)$$

Proof is simple by using block matrix. Write $\boldsymbol{b}_n = (\gamma(n), \ldots, \gamma(1))'$. Then

$$R_{n+1} = \begin{pmatrix} R_n & \boldsymbol{b}_n \\ \boldsymbol{b}_n' & \gamma(0) \end{pmatrix}.$$

The inverse of this block matrix is given by

$$R_{n+1}^{-1} = \begin{pmatrix} R_n^{-1} + c_n R_n^{-1} \boldsymbol{b}_n \boldsymbol{b}_n' R_n^{-1} & -c_n R_n^{-1} \boldsymbol{b}_n \\ -c_n \boldsymbol{b}_n' R_n^{-1} & c_n \end{pmatrix}$$

where $c_n = (\gamma(0) - \boldsymbol{b}_n' R_n^{-1} \boldsymbol{b}_n)^{-1}$.

By (3), we have

$$\boldsymbol{\phi}_{n+1} = R_{n+1}^{-1} k_{n+1}. \tag{4}$$

The proof is completed by applying the inverse given in the previous page and the fact that

$$c_n = 1/\upsilon_n, n \geq 1. \tag{5}$$

The last equation can be shown by induction.

Project Due Friday March 25

You work as a group of 3 to 4 people and turn in one report.

1. Prove equation (1).

2. Prove equation (5) and use equation (4) to complete the proof of the Durbin-Levinson algorithm. Hint: You just need to show that $c_1 = 1/v_1$ and
$$1/c_{n+1} = (1 - \phi_{nn}^2)/c_n \text{ for } n > 1.$$

3. Write a computer program to calculate the PACF of an $MA(q)$ model up to the lag of 50 by applying equation (3). You computer program should take the MA coefficients $\theta_1, \cdots, \theta_q$ as the only input and your output should be the 50 PACFs.

4. Do the same thing in the previous problem by applying the Durbin-Levinson algorithm. Report the number of lines in your code.