**4.**

**a.** $E(\overline{X}-\overline{Y}) = E(\overline{X}) - E(\overline{Y}) = \mu_1 - \mu_2$ ; $\overline{x} - \overline{y} = 8.141 - 8.575 = -.434$ .

**b.** $V(\overline{X}-\overline{Y}) = V(\overline{X}) + V(\overline{Y}) = \sigma_{\overline{X}}^2 + \sigma_{\overline{Y}}^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$ $\sigma_{\overline{X}-\overline{Y}} = \sqrt{V(\overline{X}-\overline{Y})} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$ . The estimate would be $s_{\overline{X}-\overline{Y}} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{1.66^2}{27} + \frac{2.104^2}{20}} = .5687$ .

**c.** $\frac{s_1}{s_2} = \frac{1.660}{2.104} = .7890.$

**d.** $V(X-Y) = V(X) + V(Y) = \sigma_1^2 + \sigma_2^2 = 1.66^2 + 2.104^2 = 7.1824.$

**10.**

**a.** The hint tells us that $E(\overline{X}^2) = V(\overline{X}) + [E(\overline{X})]^2$ . We know that $E(\overline{X}) = \mu$ and $SD(\overline{X}) = \frac{\sigma}{\sqrt{n}}$, so $E(\overline{X}^2) = \left(\frac{\sigma}{\sqrt{n}}\right)^2 + [\mu]^2 = \frac{\sigma^2}{n} + \mu^2$ . Since $E(\overline{X}^2) \neq \mu^2$ , we've discovered that $\overline{X}^2$ is <u>not</u> an unbiased estimator of $\mu^2$ !

In fact, the bias equals $E(\overline{X}^2) - \mu^2 = \frac{\sigma^2}{n} > 0$, so $\overline{X}^2$ is *biased high*. It will tend to *over*estimate the true value of $\mu^2$ .

**b.** By linearity of expectation, $E(\overline{X}^2 - kS^2) = E(\overline{X}^2) - kE(S^2)$ . The author proves in Section 6.1 that $E(S^2) = \sigma^2$ , so $E(\overline{X}^2) - kE(S^2) = \frac{\sigma^2}{n} + \mu^2 - k\sigma^2$ .

The goal is to find *k* so that $E(\overline{X}^2 - kS^2) = \mu^2$ . That requires $\frac{\sigma^2}{n} - k\sigma^2 = 0$ , or $k = \frac{1}{n}$ .

**11.**

**a.** $E\left(\frac{X_1}{n_1}-\frac{X_2}{n_2}\right)=\frac{1}{n_1}E(X_1)-\frac{1}{n_2}E(X_2)=\frac{1}{n_1}(n_1p_1)-\frac{1}{n_2}(n_2p_2)=p_1-p_2$.

**b.** $V\left(\frac{X_1}{n_1}-\frac{X_2}{n_2}\right)=V\left(\frac{X_1}{n_1}\right)+V\left(\frac{X_2}{n_2}\right)=\left(\frac{1}{n_1}\right)^2V(X_1)+\left(\frac{1}{n_2}\right)^2V(X_2)=$

$\frac{1}{n_1^2}(n_1p_1q_1)+\frac{1}{n_2^2}(n_2p_2q_2)=\frac{p_1q_1}{n_1}+\frac{p_2q_2}{n_2}$, and the standard error is the square root of this quantity.

**c.** With $\hat{p}_1=\frac{x_1}{n_1}$, $\hat{q}_1=1-\hat{p}_1$, $\hat{p}_2=\frac{x_2}{n_2}$, $\hat{q}_2=1-\hat{p}_2$, the estimated standard error is $\sqrt{\frac{\hat{p}_1\hat{q}_1}{n_1}+\frac{\hat{p}_2\hat{q}_2}{n_2}}$.

**d.** $(\hat{p}_1-\hat{p}_2)=\frac{127}{200}-\frac{176}{200}=.635-.880=-.245$

**e.** $\sqrt{\frac{(.635)(.365)}{200}+\frac{(.880)(.120)}{200}}=.041$

**12.** $E\left[\frac{(n_1-1)S_1^2+(n_2-1)S_2^2}{n_1+n_2-2}\right]=\frac{(n_1-1)}{n_1+n_2-2}E(S_1^2)+\frac{(n_2-1)}{n_1+n_2-2}E(S_2^2)=\frac{(n_1-1)}{n_1+n_2-2}\sigma^2+\frac{(n_2-1)}{n_1+n_2-2}\sigma^2=\sigma^2$.

**13.** $\mu=E(X)=\int_{-1}^{1}x\cdot\tfrac{1}{2}(1+\theta x)dx=\frac{x^2}{4}+\frac{\theta x^3}{6}\Big|_{-1}^{1}=\frac{1}{3}\theta\Rightarrow\theta=3\mu$

$\Rightarrow\hat{\theta}=3\overline{X}\Rightarrow E(\hat{\theta})=E(3\overline{X})=3E(\overline{X})=3\mu=3\left(\frac{1}{3}\right)\theta=\theta.$

**21.**

**a.** $E(X)=\beta\cdot\Gamma\left(1+\frac{1}{\alpha}\right)$ and $E(X^2)=V(X)+[E(X)]^2=\beta^2\Gamma\left(1+\frac{2}{\alpha}\right)$, so the moment estimators $\hat{\alpha}$ and $\hat{\beta}$ are the solution to $\bar{X}=\hat{\beta}\cdot\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$, $\frac{1}{n}\sum X_i^2=\hat{\beta}^2\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)$. Thus $\hat{\beta}=\frac{\bar{X}}{\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)}$, so once $\hat{\alpha}$ has been determined $\Gamma\left(1+\frac{1}{\hat{\alpha}}\right)$ is evaluated and $\hat{\beta}$ then computed. Since $\bar{X}^2=\hat{\beta}^2\cdot\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)$,

$\frac{1}{n}\sum\frac{X_i^2}{\bar{X}^2}=\frac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}$, so this equation must be solved to obtain $\hat{\alpha}$.

**b.** From **a**, $\frac{1}{20}\left(\frac{16{,}500}{28.0^2}\right)=1.05=\frac{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}$, so $\frac{1}{1.05}=.95=\frac{\Gamma^2\left(1+\frac{1}{\hat{\alpha}}\right)}{\Gamma\left(1+\frac{2}{\hat{\alpha}}\right)}$, and from the hint,

$\frac{1}{\hat{\alpha}}=.2\Rightarrow\hat{\alpha}=5$. Then $\hat{\beta}=\frac{\bar{x}}{\Gamma(1.2)}=\frac{28.0}{\Gamma(1.2)}$.

**22.**

**a.** $E(X)=\int_0^1 x(\theta+1)x^\theta dx=\frac{\theta+1}{\theta+2}=1-\frac{1}{\theta+2}$, so the moment estimator $\hat{\theta}$ is the solution to

$\bar{X}=1-\frac{1}{\hat{\theta}+2}$, yielding $\hat{\theta}=\frac{1}{1-\bar{X}}-2$. Since $\bar{x}=.80, \hat{\theta}=5-2=3$.

**b.** $f(x_1,\ldots,x_n;\theta)=(\theta+1)^n(x_1x_2\ldots x_n)^\theta$, so the log likelihood is $n\ln(\theta+1)+\theta\sum\ln(x_i)$. Taking $\frac{d}{d\theta}$ and equating to 0 yields $\frac{n}{\theta+1}=-\sum\ln(x_i)$, so $\hat{\theta}=-\frac{n}{\sum\ln(X_i)}-1$. Taking $\ln(x_i)$ for each given $x_i$ yields ultimately $\hat{\theta}=3.12$.

**25.**

**a.** $\hat{\mu}=\bar{x}=384.4; s^2=395.16$, so $\frac{1}{n}\sum(x_i-\bar{x})^2=\hat{\sigma}^2=\frac{9}{10}(395.16)=355.64$ and $\hat{\sigma}=\sqrt{355.64}=18.86$ (this is <u>not</u> $s$).

**b.** The 95th percentile is $\mu+1.645\sigma$, so the mle of this is (by the invariance principle) $\hat{\mu}+1.645\hat{\sigma}=415.42$.

**26.** The mle of $P(X\leq 400)$ is, by the invariance principle, $\Phi\left(\frac{400-\hat{\mu}}{\hat{\sigma}}\right)=\Phi\left(\frac{400-384.4}{18.86}\right)=\Phi(.80)=.7881$

**27.**

**a.** $f(x_1,\ldots,x_n;\alpha,\beta)=\dfrac{(x_1x_2\ldots x_n)^{\alpha-1}e^{-\Sigma x_i/\beta}}{\beta^{n\alpha}\Gamma^n(\alpha)}$, so the log likelihood is

$(\alpha-1)\sum\ln(x_i)-\dfrac{\sum x_i}{\beta}-n\alpha\ln(\beta)-n\ln\Gamma(\alpha)$. Equating both $\dfrac{d}{d\alpha}$ and $\dfrac{d}{d\beta}$ to 0 yields

$\sum\ln(x_i)-n\ln(\beta)-n\dfrac{d}{d\alpha}\Gamma(\alpha)=0$ and $\dfrac{\sum x_i}{\beta^2}=\dfrac{n\alpha}{\beta}=0$, a very difficult system of equations to solve.

**b.** From the second equation in **a**, $\dfrac{\sum x_i}{\beta}=n\alpha\Rightarrow\overline{x}=\alpha\beta=\mu$, so the mle of $\mu$ is $\hat{\mu}=\overline{X}$.

**4.**

**a.** $58.3\pm\dfrac{1.96(3)}{\sqrt{25}}=58.3\pm1.18=(57.1,59.5)$.

**b.** $58.3\pm\dfrac{1.96(3)}{\sqrt{100}}=58.3\pm.59=(57.7,58.9)$.

**c.** $58.3\pm\dfrac{2.58(3)}{\sqrt{100}}=58.3\pm.77=(57.5,59.1)$.

**d.** 82% confidence $\Rightarrow$ 1 – α = .02 $\Rightarrow$ α = .18 $\Rightarrow$ α/2 = .09, and $z_{.09}$ = 1.34. The interval is $58.3\pm\dfrac{1.34(3)}{\sqrt{100}}=(57.9,58.7)$.

**e.** $n=\left[\dfrac{2(2.58)3}{1}\right]^2=239.62\nearrow 240$.