Lecture 2

Basic Concepts and Simple Comparative Experiments

Montgomery: Chapter 2

Random Variable and probability distribution

Discrete random variable Y:

- Finite possible values $\{y_1, y_2, y_3, \dots, y_k\}$
- Probability mass function $\{p(y_1), p(y_2), \dots p(y_k)\}$ satisfying

$$p(y_i) \geq 0 \text{ and } \sum_{i=1}^{k} p(y_i) = 1.$$

Continuous random variable Y:

- Possible values form an interval
- Probability density function $f(y)$ satisfying

$$f(y) \geq 0 \text{ and } \int f(y)dy = 1.$$

Mean and Variance

Mean $\mu = \mathsf{E}(Y)$: center, location, etc.

Variance $\sigma^2 = \mathsf{Var}(Y)$: spread, dispersion, etc.

Discrete Y:

$$\mu = \sum_{i=1}^{k} y_i p(y_i); \quad \sigma^2 = \sum_{i=1}^{k} (y_i - \mu)^2 p(y_i)$$

Continuous Y:

$$\mu = \int y f(y) dy; \; \sigma^2 = \int (y - \mu)^2 f(y) dy$$

Formulas for calculating mean and variance

If Y_1 and Y_2 are **independent**, then

$\mathsf{E}(Y_1 Y_2) = \mathsf{E}(Y_1)\mathsf{E}(Y_2)$

$\mathsf{Var}(aY_1 \pm bY_2) = a^2\mathsf{Var}(Y_1) + b^2\mathsf{Var}(Y_2)$

Other formulas refer to Page 28 (Montgomery, 6th Edition)

Statistical Analysis and Inference:

Learn about population from (randomly) drawn data/sample

Model and parameter:

Assume population (Y) follows a certain model (distribution) that depends on a set of unknown constants (parameters): $Y \sim f(y, \theta)$.

Example 1: $Y \sim N(\mu, \sigma^2)$

$Y \sim \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{-\frac{(y-\mu)^2}{2\sigma^2}\}$; where $\theta = (\mu, \sigma^2)$

Example 2: Y_1 and Y_2 are yields of tomato plants fed with fertilizer mixtures A and B respectively:

$$Y_1 = \mu_1 + \epsilon_1;\ \epsilon_1 \sim N(0, \sigma_1^2)$$

$$Y_2 = \mu_2 + \epsilon_2;\ \epsilon_2 \sim N(0, \sigma_2^2)$$

$$\theta = (\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$$

Random sample or observations

Random Sample (conceptual)

$$X_1, X_2, \dots, X_n \sim f(x, \theta)$$

Random Sample (realized)

$$x_1, x_2, \dots, x_n \sim f(x, \theta)$$

Example 1:

0.0 4.9 -0.5 -1.2 2.1 2.8 1.2 0.8 0.9 -0.9

Example 2:

A: 19.6 17.9 18.0 20.3 19.3 17.1 16.7 19.2 19.9 19.3

B: 19.6 19.9 21.8 18.4 19.4 21.4 20.5 20.0 18.2 19.9

Statistical Inference: Estimating Parameter θ

- **Statistics**: a statistic is a function of the sample.
  $Y_1, \ldots, Y_n$: $\hat{\theta} = g(Y_1, Y_2, \ldots, Y_n)$ called **estimator**
  $y_1, \ldots, y_n$: $\hat{\theta} = g(y_1, y_2, \ldots, y_n)$ called **estimate**

- **Example 1**:
  Estimators for μ and σ^2

$$\hat{\mu} = \bar{Y} = \frac{\sum_{i=1}^n Y_i}{n};\ \hat{\sigma}^2 = S^2 = \frac{\sum_{i=1}^n (Y_i - \bar{Y})^2}{n-1}$$

  Estimates

$$\hat{\mu} = \bar{y} = 1.01; \hat{\sigma}^2 = s^2 = 3.49$$

- **Example 2**:

  Estimators:

$$\hat{\mu}_i = \bar{Y}_i = \frac{\sum_{j=1}^{n_i} Y_{ij}}{n_i};\ \hat{\sigma}_i^2 = S_i^2 = \frac{\sum_{j=1}^{n_i}(Y_{ij} - \bar{Y}_i)^2}{n_i - 1}$$

  for $i = 1, 2$.

  Estimates:

$$\bar{y}_1 = 18.73; s_1^2 = 1.50; \bar{y}_2 = 19.91; s_2^2 = 1.30;$$

  Assume $\sigma_1^2 = \sigma_2^2$:

$$S_{pool}^2 = \frac{(n_1 - 1)S_1^2 + (n_2 - 1)S_2^2}{n_1 + n_2 - 2};\ s_{pool}^2 = 1.40$$

Statistical Inference: Testing Hypotheses

Use test statistics and their distributions to judge hypotheses regarding parameters.

- **H_0: null hypothesis vs H_1: alternative hypothesis**

  Example 1: $H_0 : \mu = 0$ vs $H_1 : \mu \neq 0$

  Example 2.1: $H_0 : \mu_2 = \mu_1$ vs $H_1 : \mu_2 > \mu_1$

  Example 2.2: $H_0 : \sigma_1^2 = \sigma_2^2$ vs $H_1 : \sigma_1^2 \neq \sigma_2^2$

  Details refer to Table 2-3 on Page 47 and Table 2-7 on Page 53

- **Test statistics:**

  Measures the amount of deviation of estimates from H_0

  Example 1:

$$T = \frac{\bar{Y} - 0}{S/\sqrt{n}} \quad \sim^{H_0} \quad t(n-1); \qquad T_{obs} = 1.71$$

Example 2:

$$T = \frac{(\bar{Y}_2 - \bar{Y}_1) - 0}{S_{pool}\sqrt{\frac{1}{n_1} + \frac{1}{n_2}}} \quad \sim^{H_0} \quad t(n_1 + n_2 - 2); \qquad T_{obs} = 2.22$$

- **Decision Rules**
  - Given significance level α, there are two approaches:
  - Compare observed test statistic with critical value
  - Compute the P-value of observed test statistic
    - Reject H_0, if the P-value $\leq \alpha$.

Statistical Inference: Testing Hypotheses

- P**-value** is the probability that test statistic takes on a value that is **at least as extreme as** the observed value of the statistic when H_0 is true.

  "Extreme" in the sense of the alternative hypothesis H_1.

  Example 1:

  $$P - \text{value} = \mathsf{P}(T \leq -1.71 \text{ or } T \geq 1.71 \mid t(9)) = .12$$

  Conclusion: fail to reject H_0 because 12% $\geq$ 5%.

  Example 2:

  $$P - \text{value} = \mathsf{P}(T \geq 2.22 \mid t(18)) = 0.02$$

  Conclusion: reject H_0 because 2% $\leq$ 5%.

Type I Error, Type II Error and Power of a Decision Rule

Type I error: when H_0 is true, reject H_0.

$$\alpha = P(\text{type I error}) = P(\text{reject } H_0 \mid H_0 \text{ is true})$$

Type II error: when H_0 is false, not reject H_0.

$$\beta = P(\text{type II error}) = P(\text{not reject } H_0 \mid H_0 \text{ is false})$$

Power

$$\text{Power} = 1 - \beta = P(\text{reject } H_0 \mid H_0 \text{ is false})$$

Details refer to Chapter 2, Stat511, etc.

In testing hypotheses, we usually control α (the significance level) and prefer decision rules with small β (or high power). Requirements on β (or power) are usually used to calculate necessary sample size.

Statistical Inference: Confidence Intervals:

Interval statements regarding parameter θ

100(1-α) percent confidence interval for θ: (L, U)

Both L and U are statistics (calculated from a sample), such that

$$P(L < \theta < U) = 1 - \alpha$$

Given a real sample $x_1, x_2, \ldots, x_n$, $l = L(x_1, \ldots, x_n)$ and $u = U(x_1, \ldots, x_n)$ lead to a confidence interval (l, u).

Question:

$$P(l < \theta < u) = ?$$

Example 1.

A 95% Confidence Interval for μ:

$$(L, U) = (\overline{Y} - t_{0.025}(9)\frac{S}{\sqrt{n}}, \overline{Y} + t_{0.025}(9)\frac{S}{\sqrt{n}})$$

For the given sample;

$$(l, u) = (1.01 - 2.26 * \frac{1.87}{\sqrt{10}}, 1.01 + 2.26 * \frac{1.87}{\sqrt{10}}) = (-.33, 2.35)$$

Example 2.

A 95% Confidence interval for $\mu_2 - \mu_1$:

$$(L, U) = \overline{Y}_2 - \overline{Y}_1 \pm t_{0.025}(18)S_{pool}\sqrt{1/n_1 + 1/n_2}$$

$$(l, u) = (19.91 - 18.83) \pm 2.10 * 1.18 * \sqrt{1/10 + 1/10} = (.07, 2.29)$$

Connection between two-sided hypothesis testing and C.I.

If the C.I. contains zero, fail to reject H_0; otherwise, reject H_0.

Sampling Distributions

Distributions of statistics used in estimation, testing and C.I. construction

Random sample: $Y_1, Y_2, \ldots, Y_n \sim N(\mu, \sigma^2)$

Sample mean $\overline{Y} = (Y_1 + Y_2 + \cdots + Y_n)/n$

$$E(\overline{Y}) = E(\frac{1}{n}\sum Y_i) = \frac{1}{n}\sum E(Y_i) = \frac{1}{n}n\mu = \mu$$

$$\text{Var}(\overline{Y}) = \text{Var}(\frac{1}{n}\sum Y_i) = \frac{1}{n^2}\sum \text{Var}(Y_i) = \frac{1}{n^2}n\sigma^2 = \sigma^2/n$$

$\overline{Y}$ follows $N(\mu, \frac{\sigma^2}{n})$

The Central Limit Theorem

$Y_1, Y_2, \ldots, Y_n$ are n independent and identically distributed random variables with $E(Y_i) = \mu$ and Var$(Y_i) = \sigma^2$. Then

$$Z_n = \frac{\overline{Y} - \mu}{\sigma/\sqrt{n}}$$

has an approximate $N(0,1)$ distribution.

Remark

1. Do not need to assume that the population distribution is normal
2. When the population distribution is normal, then Z_n exactly follows $N(0,1)$.

Sampling Distributions: Sample Variance

$$S^2 = \frac{(Y_1 - \overline{Y})^2 + (Y_2 - \overline{Y})^2 + \cdots + (Y_n - \overline{Y})^2}{n-1}$$

$$E(S^2) = \sigma^2$$

$$\frac{(n-1)S^2}{\sigma^2} = \frac{\sum_{i=1}^{n}(Y_i - \overline{Y})^2}{\sigma^2} \sim \chi^2_{n-1}$$

Chi-squared distribution

If Z_1, Z_2, . . . , Z_k are i.i.d as $N(0, 1)$, then

$$W = Z_1^2 + Z_2^2 + \cdots + Z_k^2$$

follows a Chi-squared distribution with degree of freedom k, denoted by χ^2_k

Density functions of χ_k^2

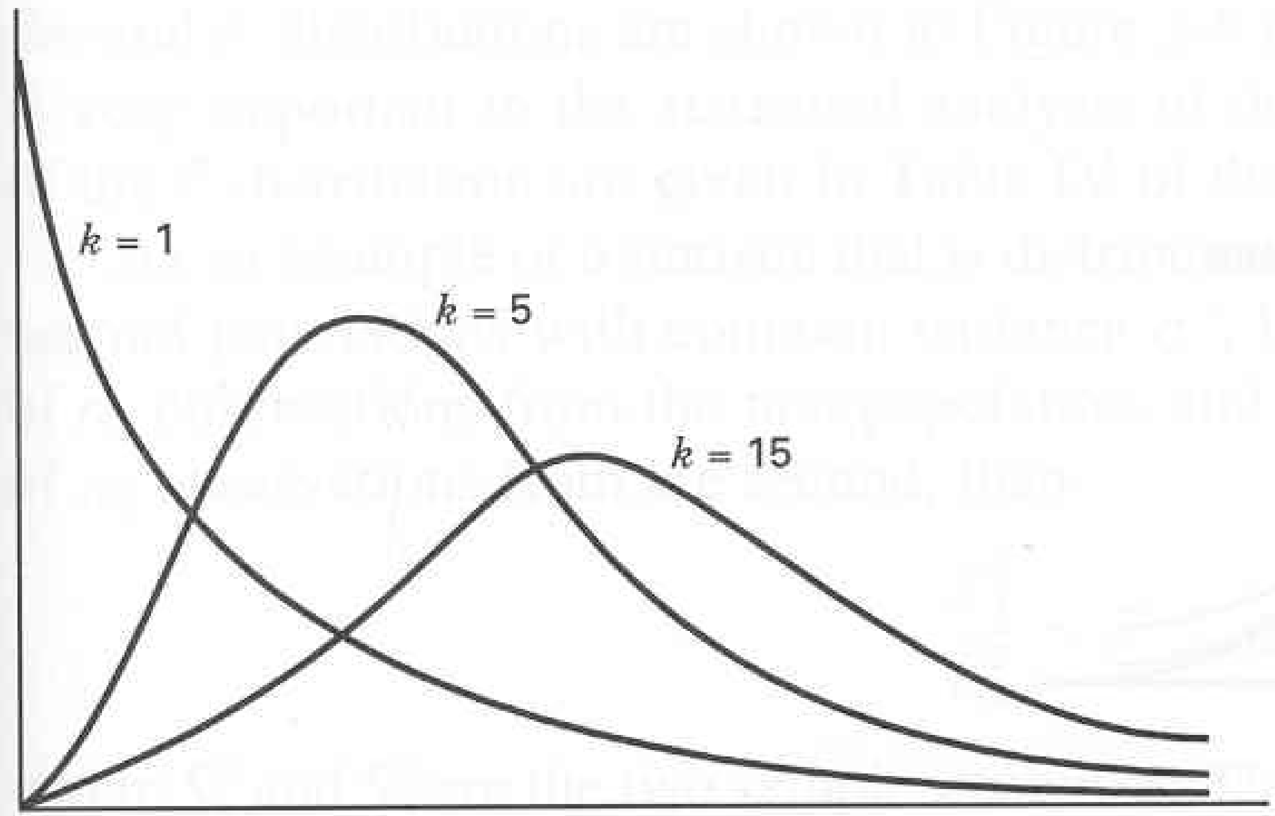


Figure 2-6 Several chi-square distributions.

Sampling Distributions

- **t-distribution: $t(k)$**

  If $Z \sim N(0,1)$, $W \sim \chi_k^2$ and Z and W independent, then

  $$T_k = \frac{Z}{\sqrt{W/k}}$$

  follows a t-distribution with d.f. k, i.e., $t(k)$.

  For example, in t-test:

  $$T = \frac{\overline{Y} - \mu_0}{S/\sqrt{n}} = \frac{\sqrt{n}(\overline{Y} - \mu_0)/\sigma}{\sqrt{S^2/\sigma^2}} = \frac{Z}{\sqrt{W/(n-1)}} \sim t(n-1)$$

  Remark:

  As n goes to infinity, $t(n-1)$ converges to $N(0,1)$.

Density functions of $t(k)$ distributions

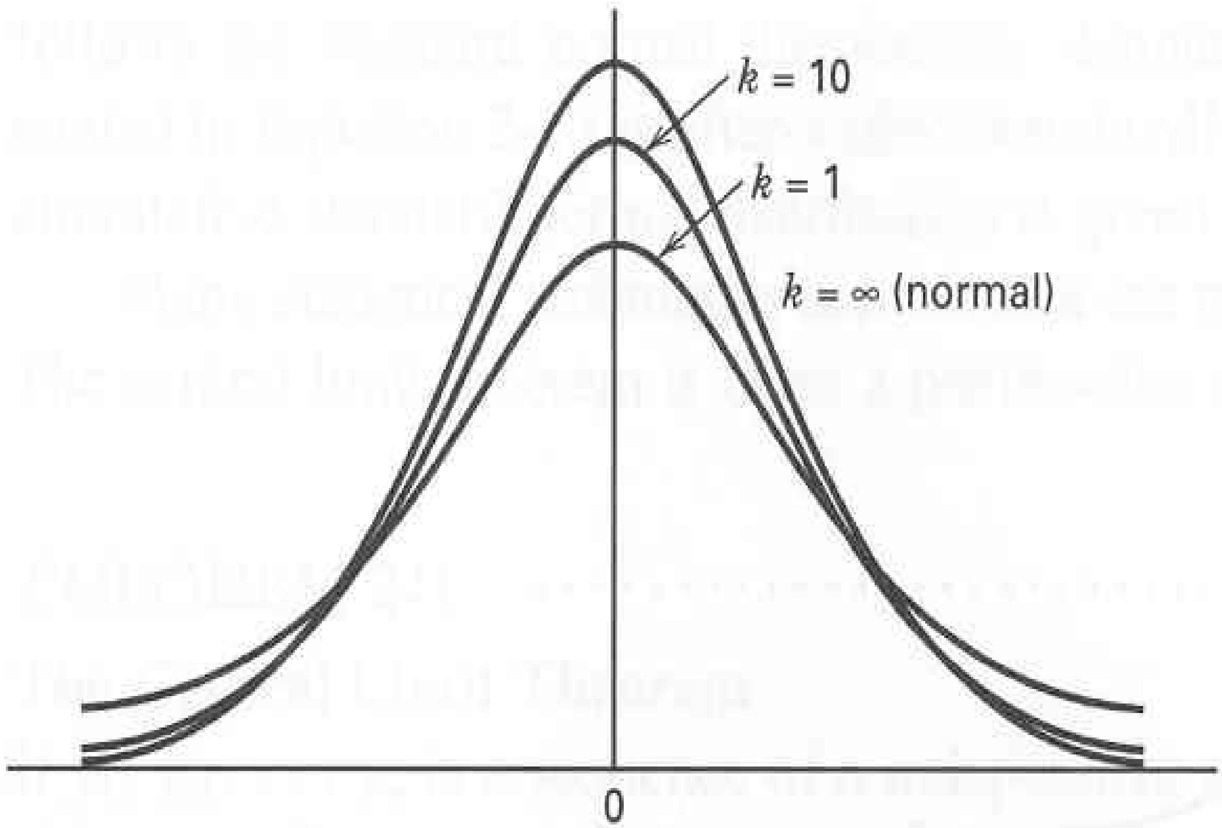


Figure 2-7 Several *t* distributions.

Sampling Distribution

- **F-distributions:** F_{k_1,k_2}

  Suppose random variables $W_1 \sim \chi^2_{k_1}$, $W_2 \sim \chi^2_{k_2}$, and W_1 and W_2 are independent, then

  $$F = \frac{W_1/k_1}{W_2/k_2}$$

  follows F_{k_1,k_2} with numerator d.f. k_1 and denominator d.f. k_2.

- **Example:** $H_0 : \sigma_1^2 = \sigma_2^2$, the test statistic is

  $$F = \frac{S_1^2}{S_2^2} = \frac{S_1^2/\sigma^2}{S_2^2/\sigma^2} = \frac{W_1/(n_1-1)}{W_2/(n_2-1)} \sim F_{n_1-1,n_2-1}$$

  Refer to Section 2.6 for details.

Density functions of F-distributions

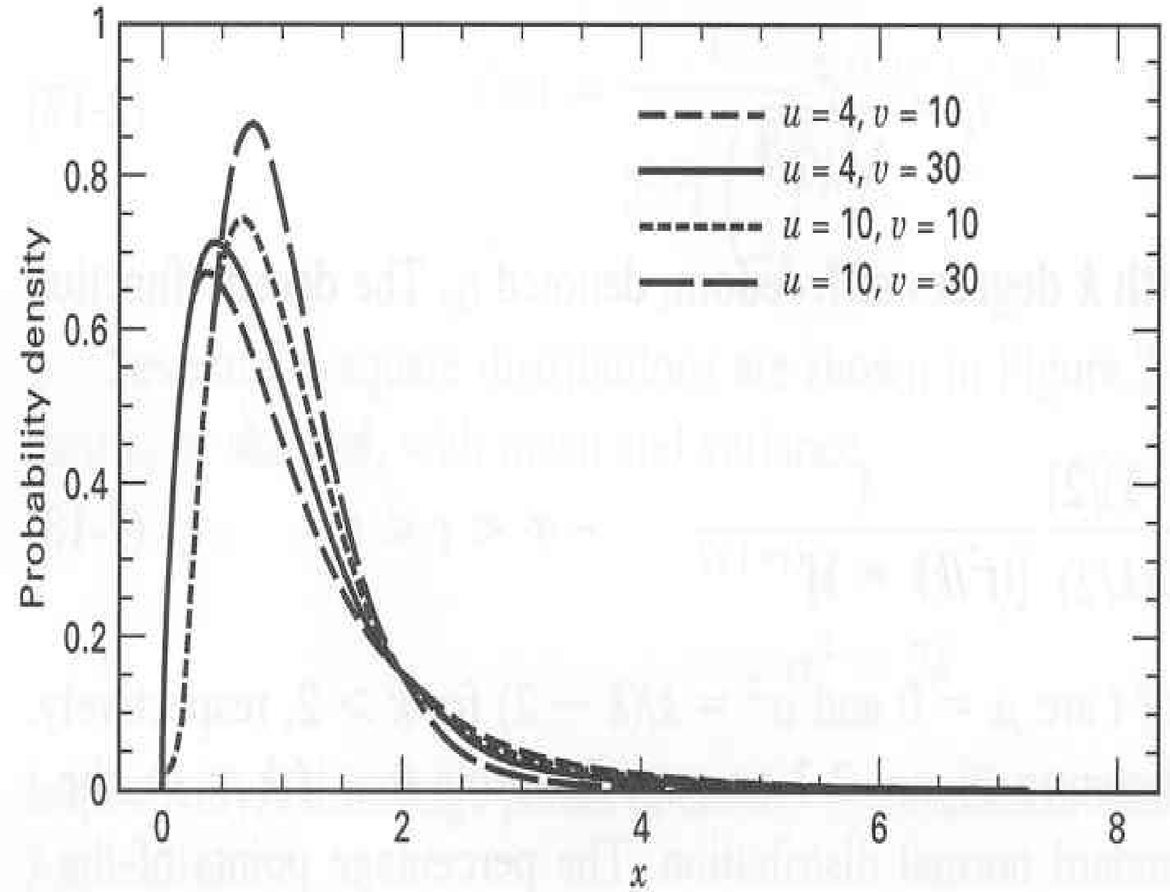


Figure 2-8 Several F distributions.

Normal Probability Plot

used to check if a sample is from a normal distribution

$Y_1, Y_2, \ldots, Y_n$ is a random sample from a population with mean μ and variance σ^2.

Order Statistics: $Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)}$ where $Y_{(i)}$ is the ith smallest value.

if the population is normal, i.e., $N(\mu, \sigma^2)$, then

$E(Y_{(i)}) \approx \mu + \sigma r_{\alpha_i}$ with $\alpha_i = \frac{i-3/8}{n+1/4}$
where r_{α_i} is the 100α_i th percentile of $N(0,1)$ for $1 \leq i \leq n$.

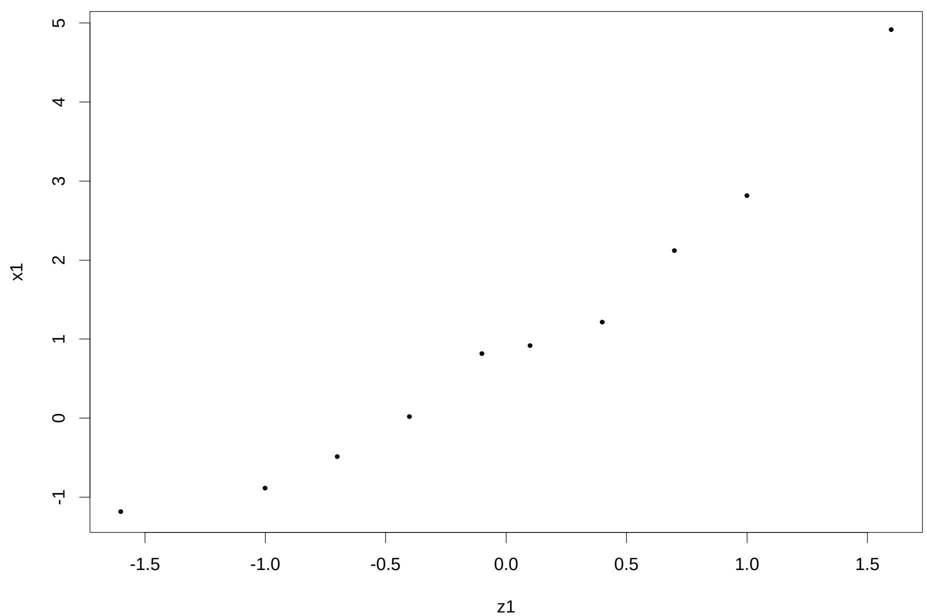
Given a sample $y_1, y_2, \ldots, y_n$, the plot of $(r_{\alpha_i}, y_{(i)})$ is called the normal probability plot or QQ plot.

the points falling around a straight line indicate normality of the population; Deviation from a straight line pattern indicates non-normality (the pen rule)

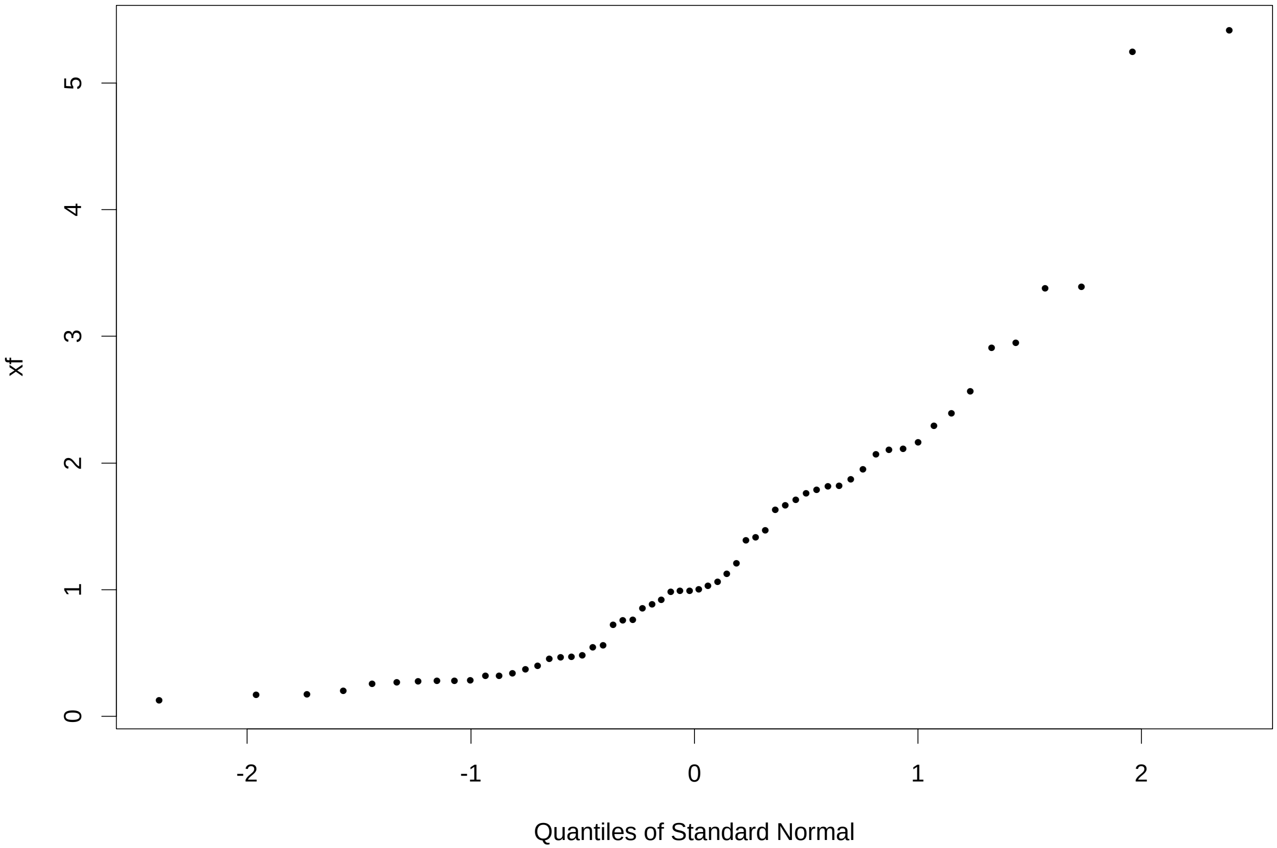
Example 1

$y_{(i)}$	-1.2	-0.9	-0.5	0.0	0.8	0.9	1.2	2.1	2.8	4.9
α_i	.06	.16	.26	.35	.45	.55	.65	.74	.84	.94
r_{α_i}	-1.6	-1.0	-.7	-.4	-.1	.1	.4	.7	1.0	1.6

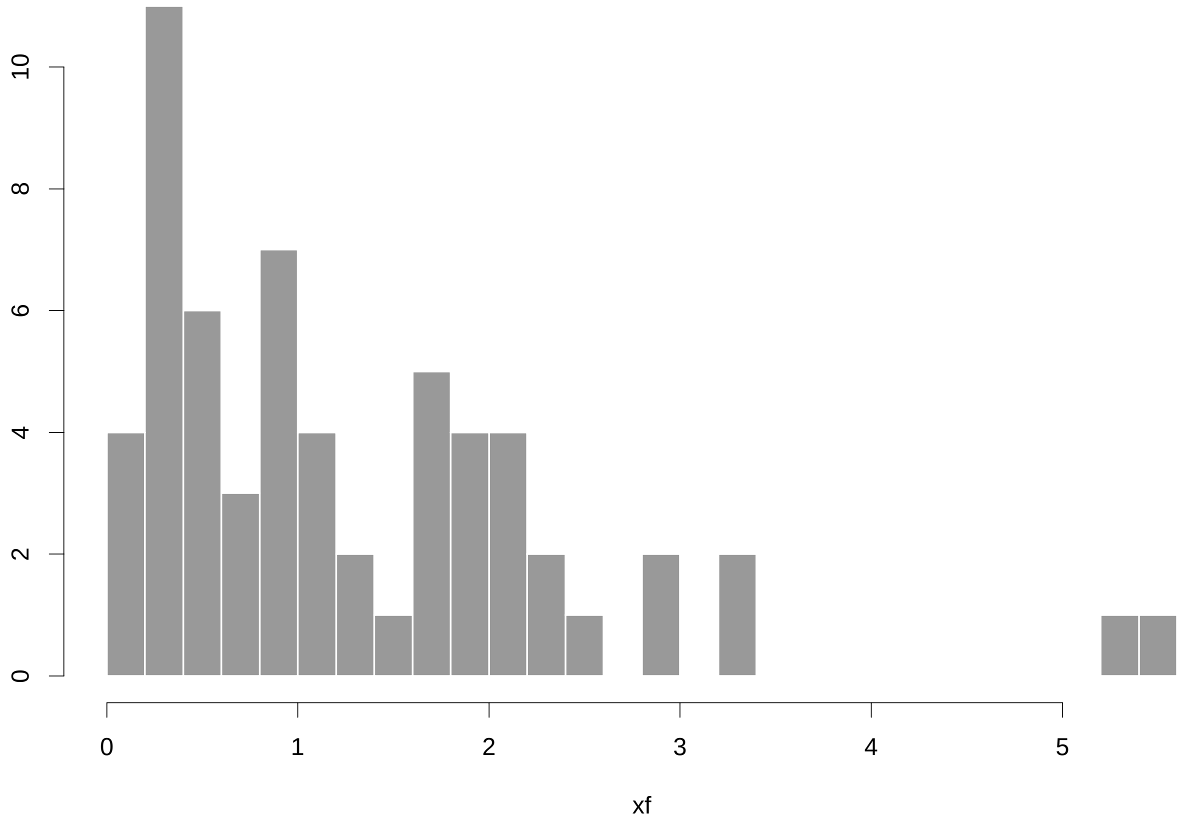
Note: r_{α_i} were obtained from the Z-chart (table)



QQ Plot 1

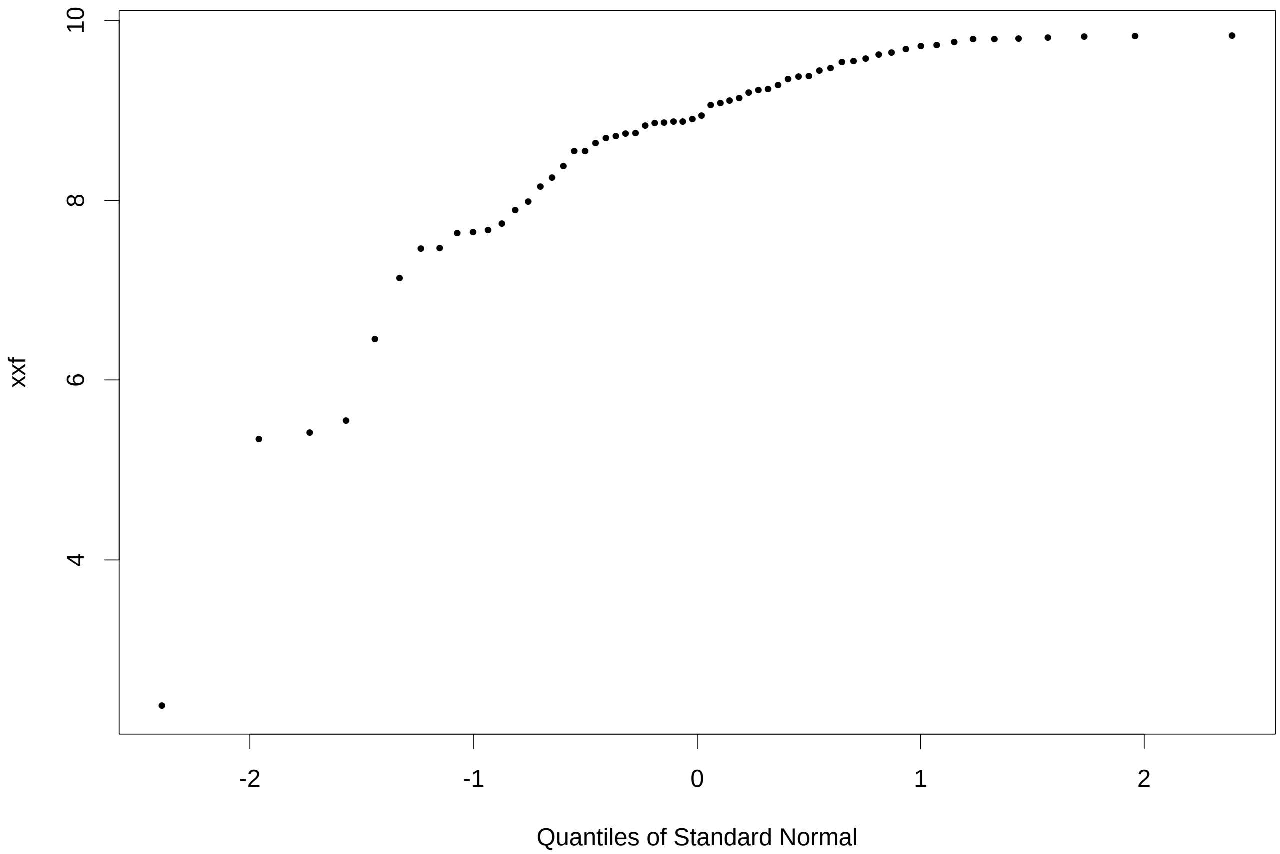


QQ Plot 1 (continued): True Population Distribution

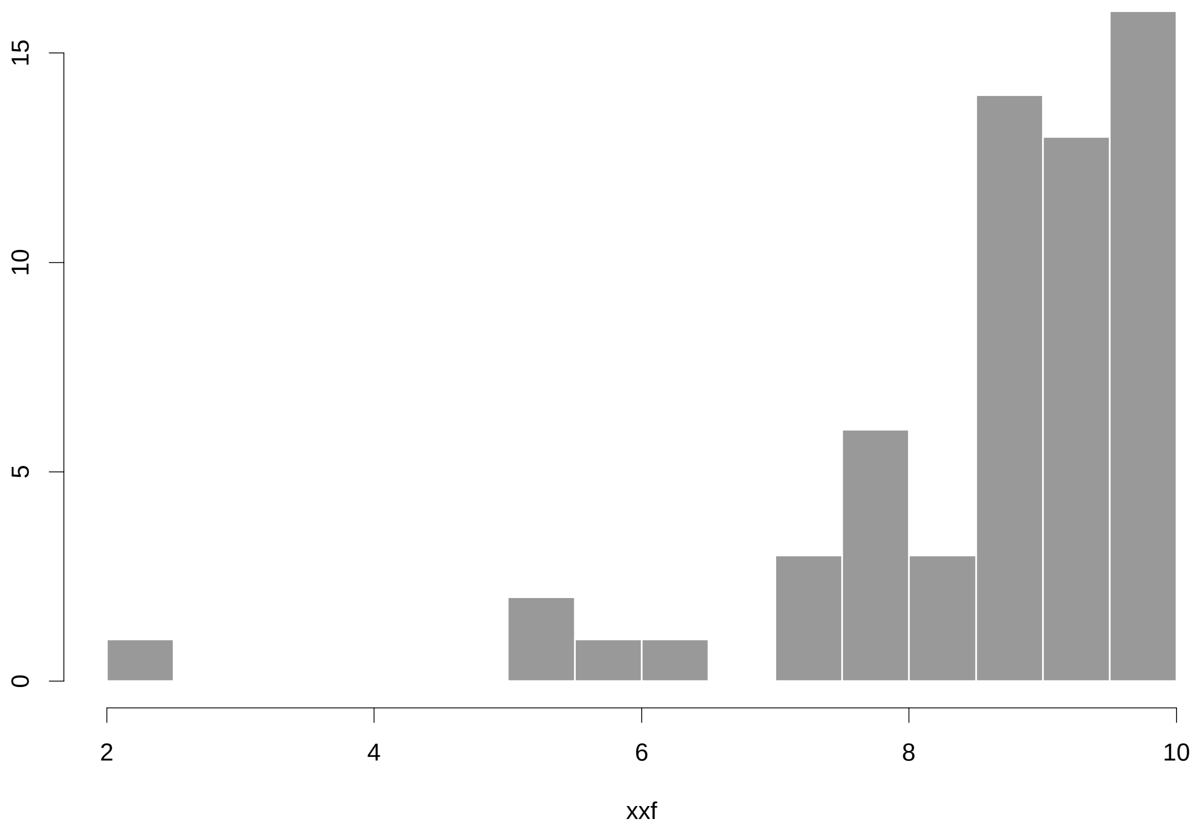


Concave-upward shape indicates right-skewed distn

QQ plot 2.

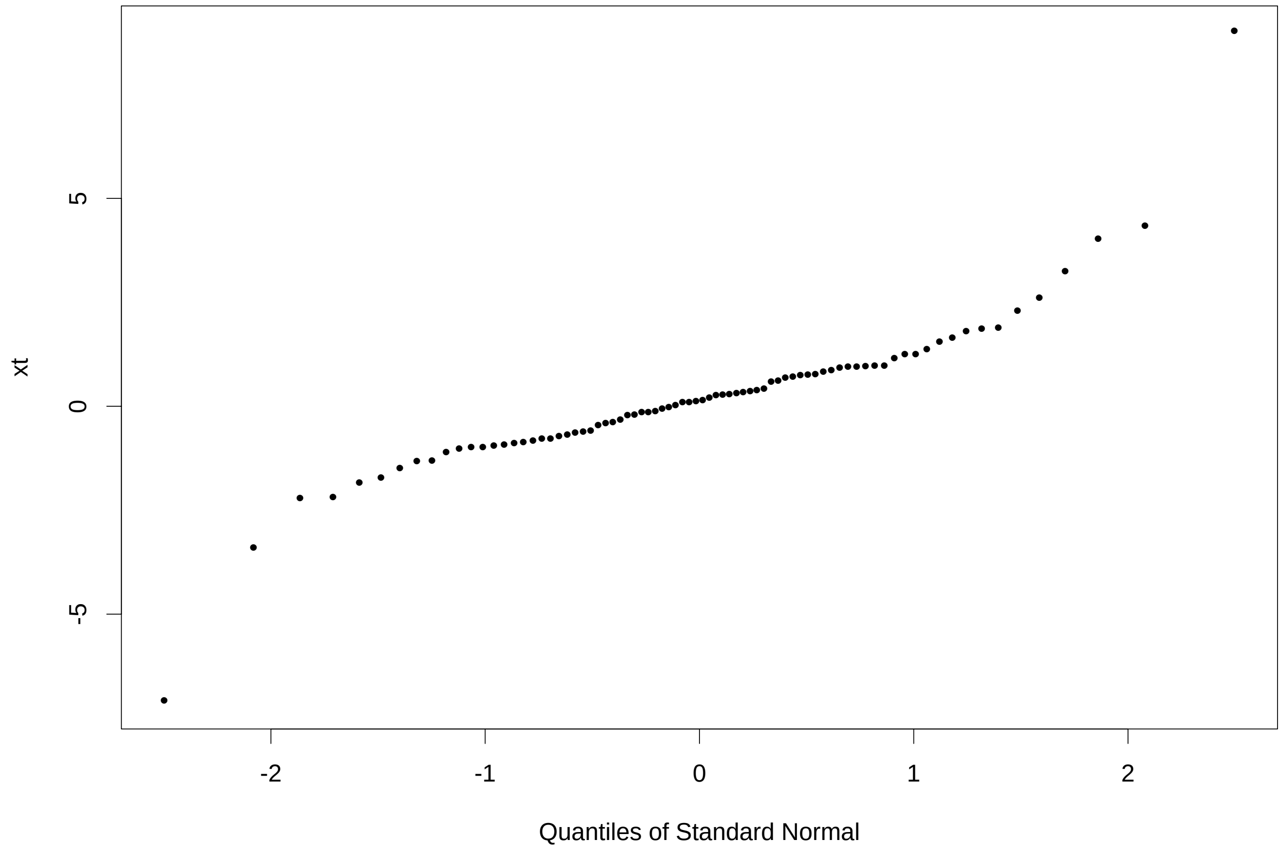


QQ plot 2 (continued)

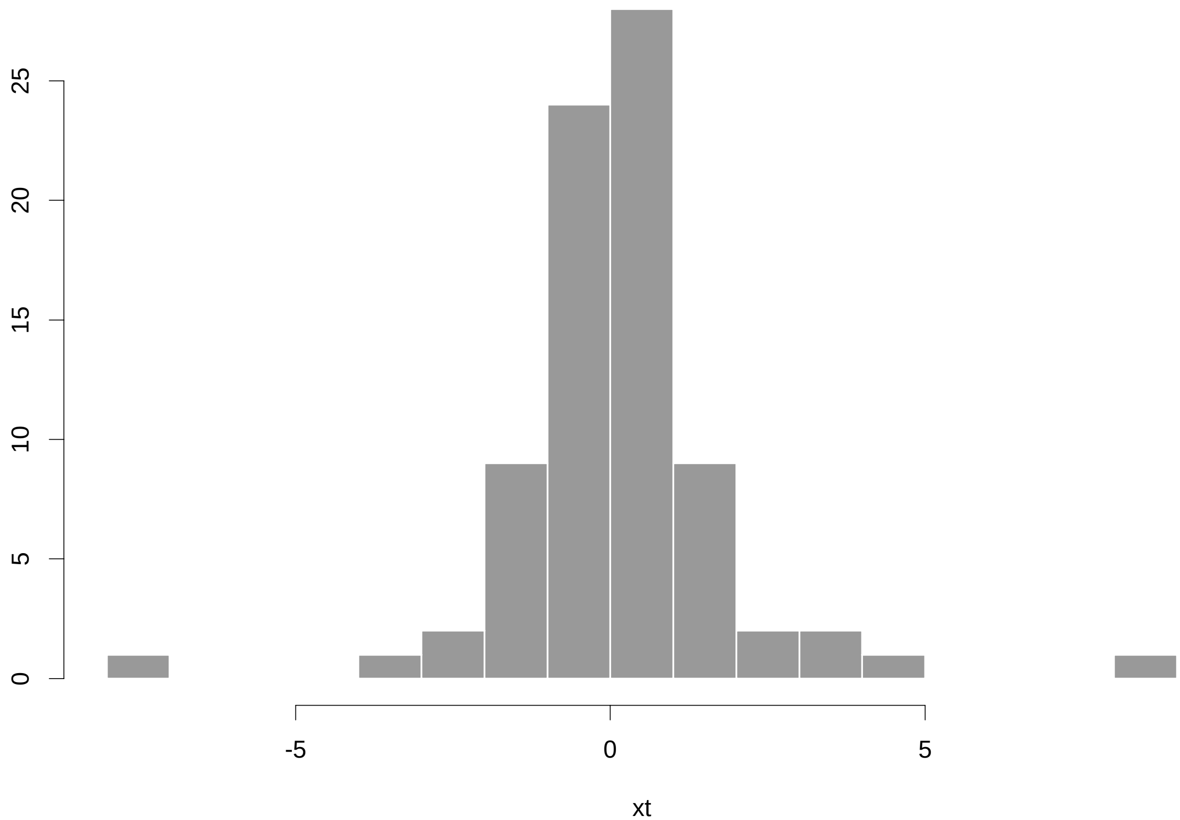


Concave-downward shape indicates left-skewed distn

QQ plot 3.



QQ plot 3 (continued)



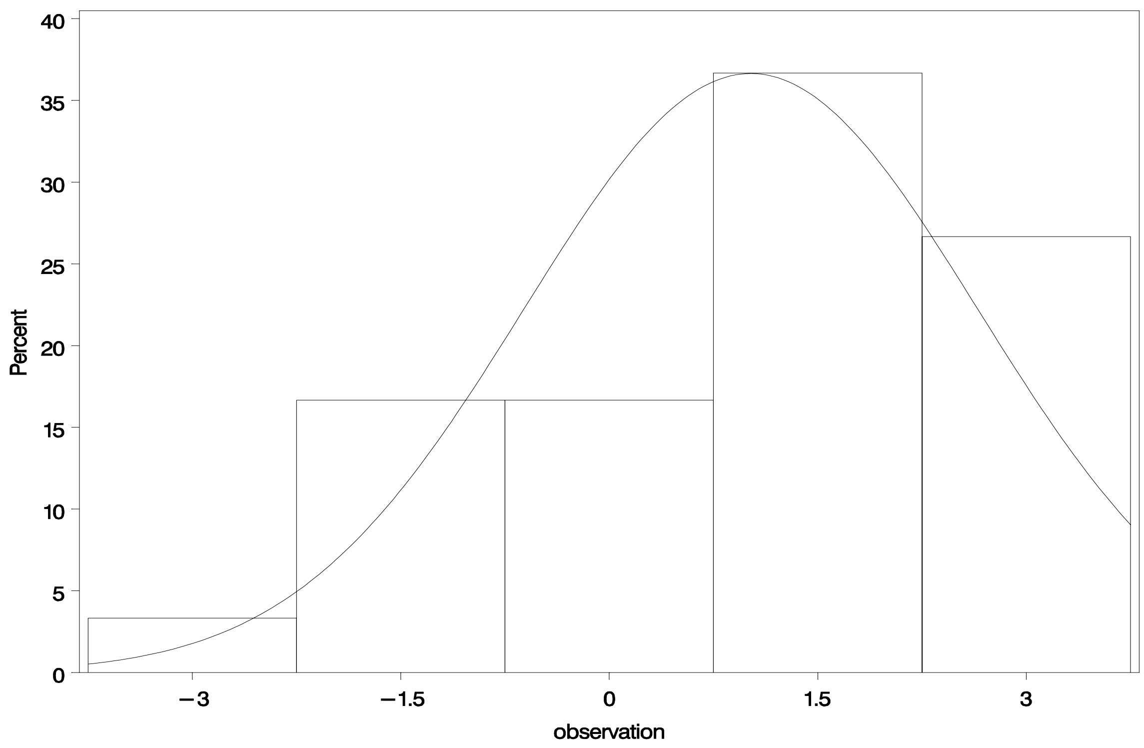
flipped S shape indicates a distribution with two heavier tails

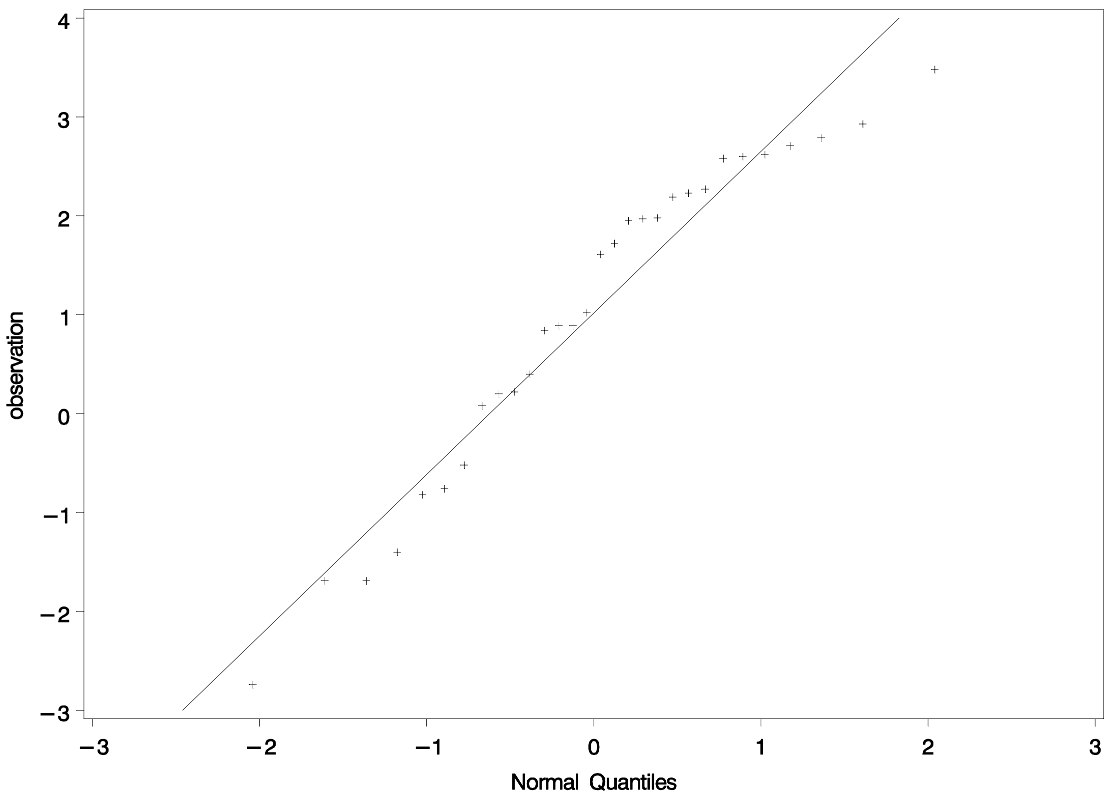
SAS Code for QQ plot

```
data one;
input observation @@;
datalines;
0.89  2.79  2.27  2.58  1.72  2.93 -0.82 -1.40  0.08  1.97
0.84 -2.74  2.62  3.48  1.95  2.23  1.02 -0.76  0.20 -1.69
-1.69  0.89  1.98  1.61 0.22  2.60 -0.52  0.40  2.71  2.19
;

proc univariate data=one;
var observation;
histogram observation / normal;
qqplot observation /normal (L=1 mu=est sigma=est);
run;
quit;
```

Output




observation
Normal Quantiles
4
3
2
1
0
−1
−2
−3
−3
−2
−1
0
1
2
3

Choice of Sample Size

- **Type II error: β=P(fail to reject $H_0 \mid H_1$ is correct)**

  In testing hypotheses, one first wants to control type I error. If type II error is too large, the conclusion would be too conservative.

- **Example 2** $H_0 : \mu_2 - \mu_1 = 0$ vs $H_1 : \mu_2 - \mu_1 \neq 0$
  - Significance level: $\alpha = 5\%$
  - For convenience, we assume two samples have the same size n
  - Decision Rule based on two-sample t-test:

    reject $H_0,$ if $\frac{\overline{Y}_2 - \overline{Y}_1}{S_{pool}\sqrt{1/n+1/n}} > t_{0.025}(2n-2)$ or $< -t_{0.025}(2n-2)$

    Equivalently

    fail to reject H_0 if $-t_{0.025}(2n-2) \leq \frac{\overline{Y}_2 - \overline{Y}_1}{S_{pool}\sqrt{1/n+1/n}} \leq t_{0.025}(2n-2)$

  The type I error of the decision rule is 5%, we want to know how large n should be so that the decision rule has type II error less than a threshold, say, 5%.

Recall

$$\beta = P(\text{type II}) = P(\text{ accept } H_0 | H_1 \text{ holds })$$

Hence

$$\beta = P(-t_{0.025}(2n-2) \leq \frac{\overline{Y}_2 - \overline{Y}_1}{S_{pool}\sqrt{1/n + 1/n}} \leq t_{0.025}(2n-2) \mid H_1)$$

Under H_1, the test statistic does not follow $t(2n-2)$, in fact, it follows a noncentral t-distribution with df $2n-2$ and noncentral parameter $\delta = \frac{|\mu_2 - \mu_1|}{\sigma\sqrt{2/n}}$. Hence β is a function of $|\mu_2 - \mu_1|/2\sigma$, and n,

$$\beta = \beta(|\mu_2 - \mu_1|/2\sigma, n)$$

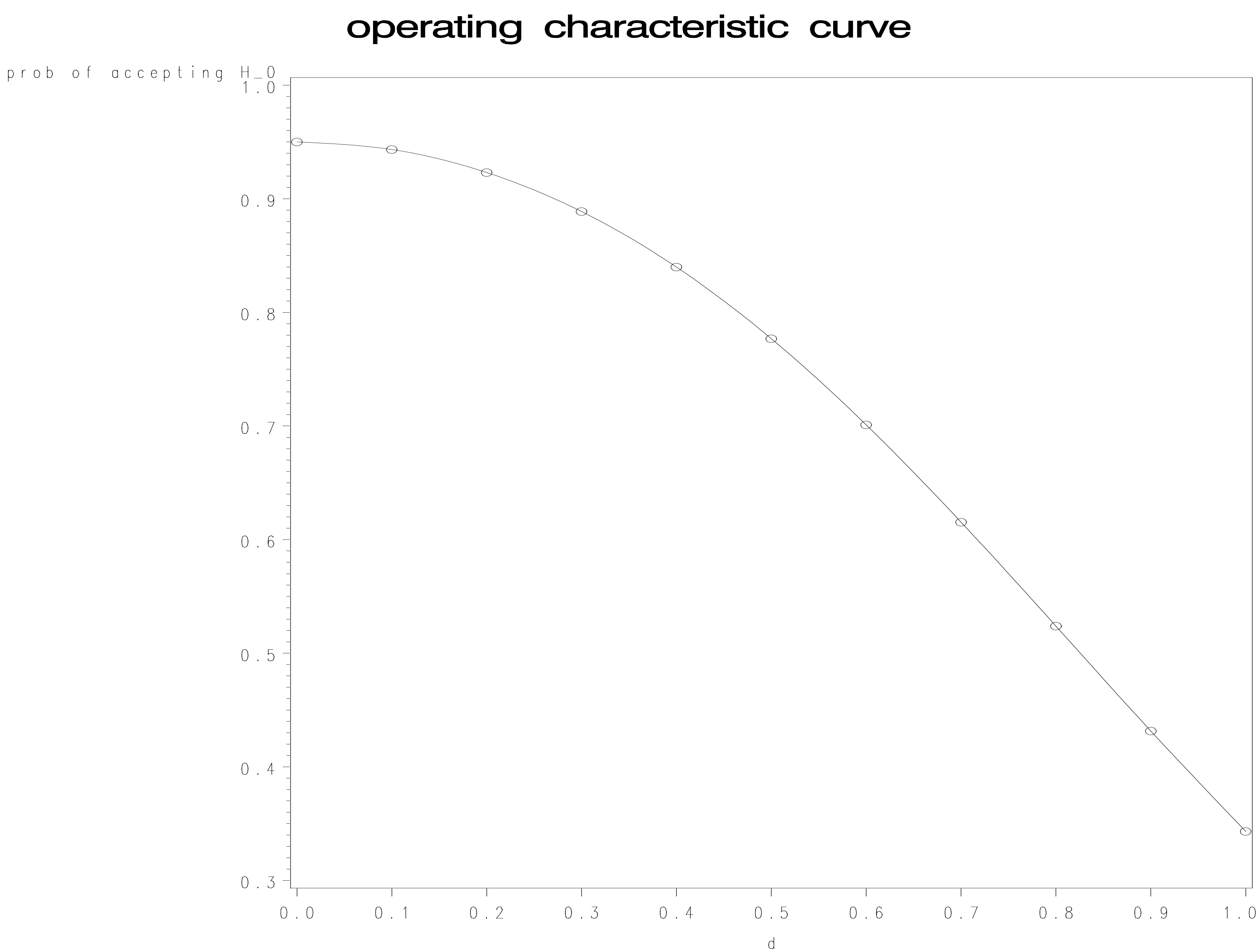
Choice of Sample Size (continued)

- Let $d = \frac{|\mu_2 - \mu_1|}{2\sigma}$. So $\beta = \beta(d, n)$, which is the probability of type II error when μ_1 and μ_2 are apart by d. Intuitively, the smaller d is, the larger n needs to be such that $\beta \le 5\%$.
- In terms of power (1-$\beta(d, n)$). The smaller d is, the larger n needs to be in order to detect μ_1 and μ_2 are different from each other.
- Suppose we are interested in making the correct decision when μ_1 and μ_2 are apart by at least $d = 1$ with high probability (power), that is, we want to guarantee the type II error at $d = 1$, $\beta(1, n)$ to be small enough, say $< 5\%$. How many data points we need to collect?:

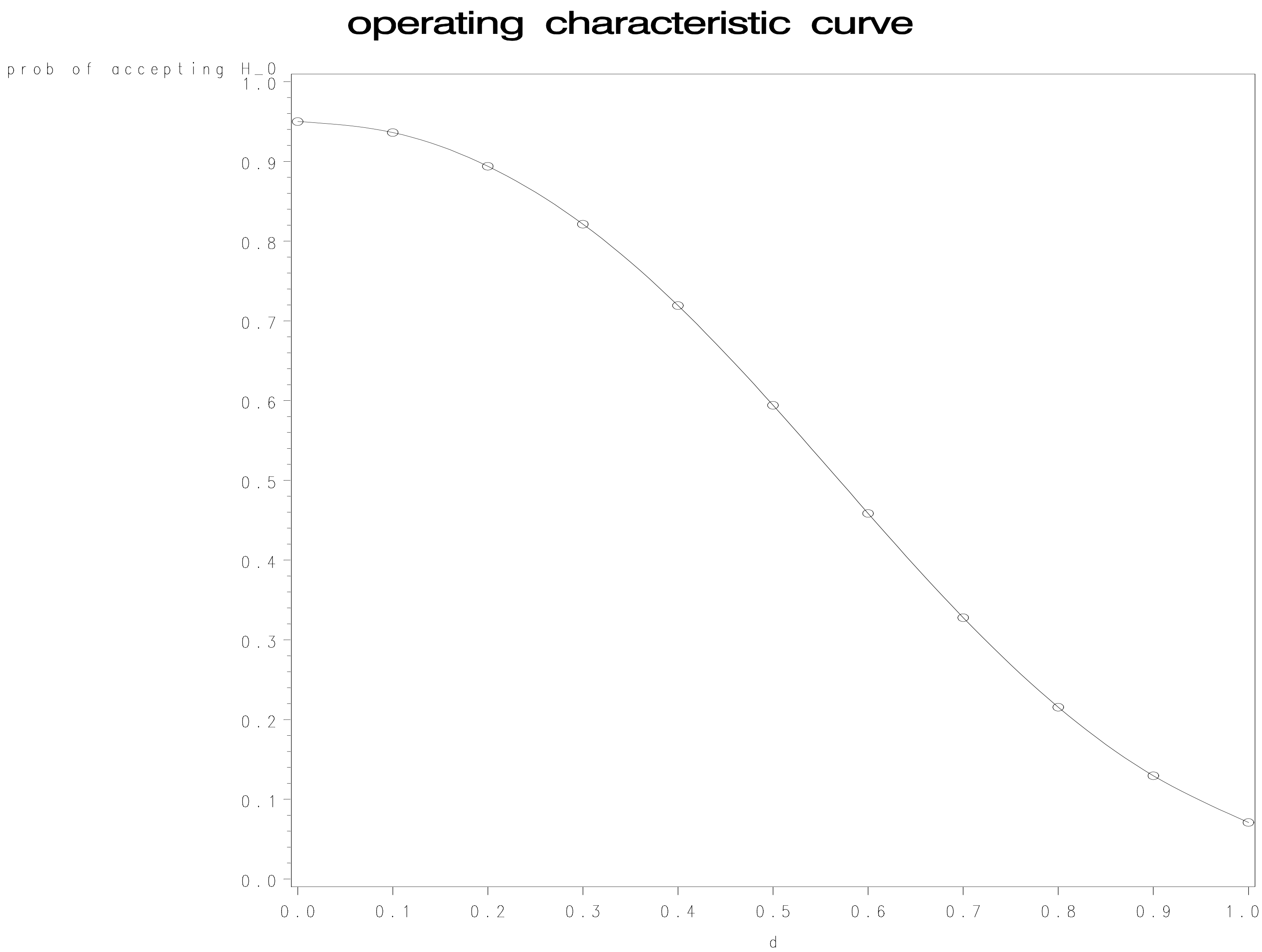
  Find the smallest n such that $\beta(1, n) < 5\%$

- Calculate $\beta(d, n)$ for $d > 0$ and fixed n and plot $\beta(d, n)$ against d, until the smallest n is found.

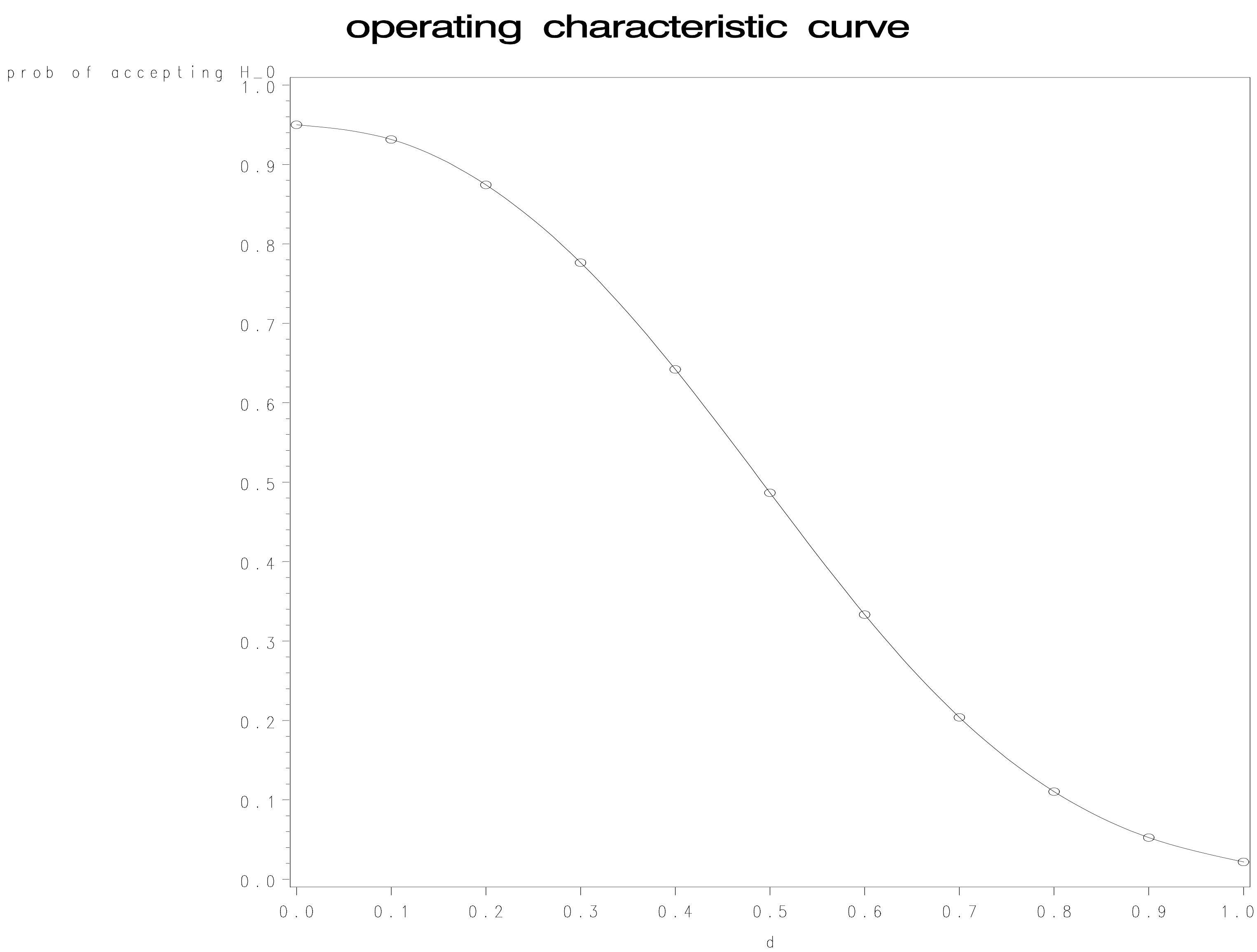
- **Case 1: n=4**


operating characteristic curve
prob of accepting H_0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
d

Case 2: n=7


operating characteristic curve
prob of accepting H_0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
d

Case 3: n=9


operating characteristic curve
prob of accepting H_0
1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
0.0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
d

Operating characteristic Curves

- Curves of $\beta(d, n)$ versus d for various given n are called operating characteristic curves, **O.C. Curves**, which can be used to determine sample size
- O.C. Curves for two-sided t test (next slide)
- $n = n_1 + n_2 - 1$. From the curves,

$$n_1 + n_2 - 1 \approx 16$$

If equal sample size is required, then $n_1 = n_2 \approx 9$.

O.C. Curves for two-sided t test

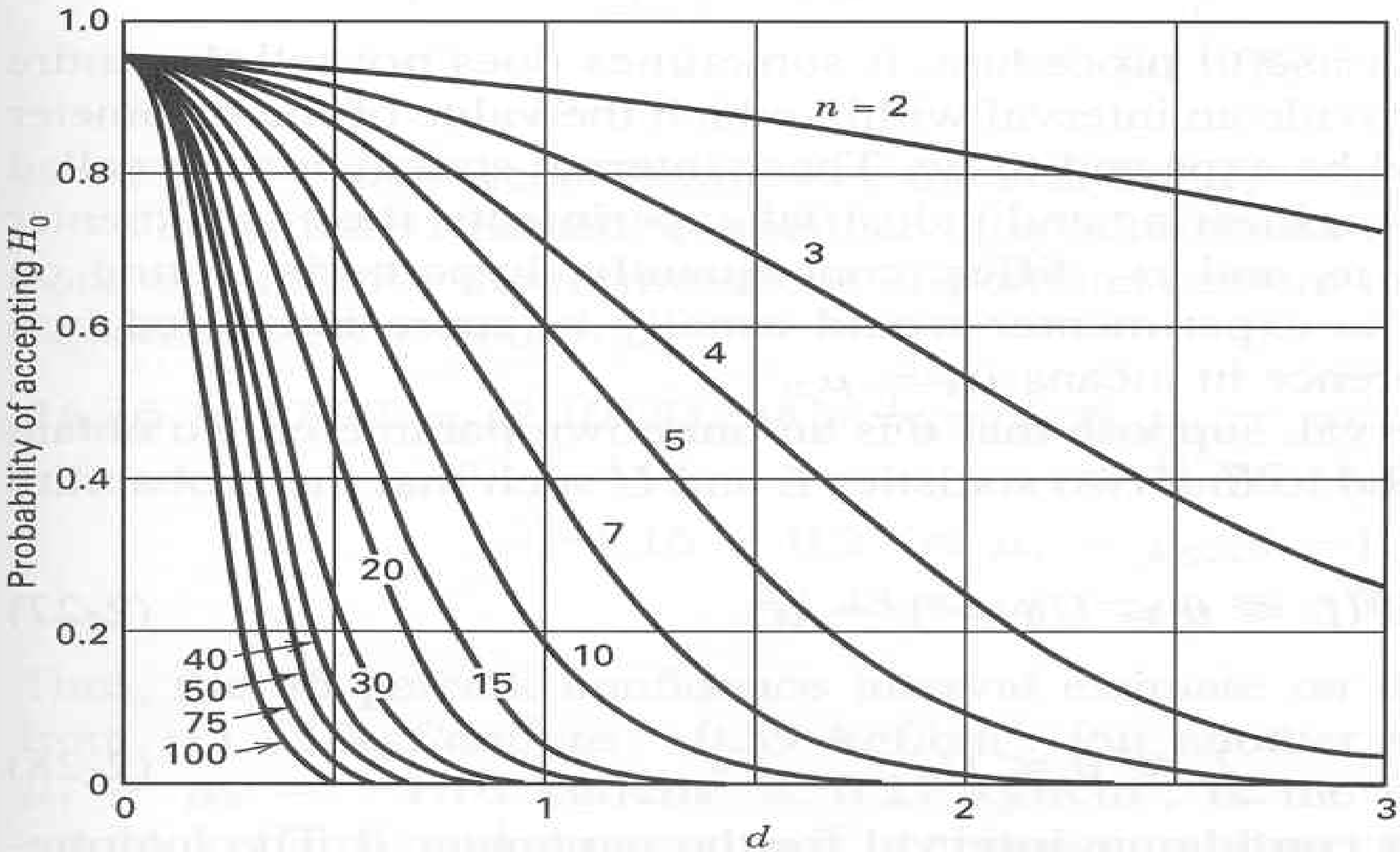


Figure 2-12 Operating characteristic curves for the two-sided t-test with $\alpha = 0.05$. (Reproduced with permission from "Operating Characteristics for the Common Statistical Tests of Significance," C. L. Ferris, F. E. Grubbs, and C. L. Weaver, *Annals of Mathematical Statistics*, June 1946.)

SAS code for plotting O.C. Curves

```
data one;
n=9;df=2*(n-1);alpha=0.05;
do d=0 to 1 by 0.10;
nc=d*sqrt(2*n);
rlow=tinv(alpha/2,df); rhigh=tinv(1-alpha/2,df);
p=probt(rhigh,df,nc)-probt(rlow,df,nc);
output;
end;

proc print data=one;
symbol1 v=circle i=sm5;
title1 'operating characteristic curve';
axis1 label=('prob of accepting H_0'); axis2 label=('d');

proc gplot;
plot p*d/haxis=axis2 vaxis=axis1;
run;
quit;
```