# Homework 5 Solution

## Problem 1

(a) Let's look at the ANOVA table output from SAS.

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 3 | 6026.83333 | 2008.94444 | 6.97 | 0.0022 |
| Error | 20 | 5767.00000 | 288.35000 | | |
| Corrected Total | 23 | 11793.83333 | | | |

The $p$-value for the $F$-statistic is 0.0022, much less than 0.05. Hence we reject the hypothesis and conclude that there are treatment differences.

(b) Since the experiment is a balanced design, two contrasts are orthogonal to each other iff their inner product is 0. Let $\mathbf{C}_1$, $\mathbf{C}_2$ and $\mathbf{C}_3$ be respectively the contrasts for "Hormone I vs Hormone II", "Low Level vs High Level" and "Equivalence of Level". Then

$$\begin{aligned}
\mathbf{C}_1^t\mathbf{C}_2 &= 1*1+1*(-1)+(-1)*1+(-1)*(-1) = 1-1-1+1 = 0,\\
\mathbf{C}_1^t\mathbf{C}_3 &= 1*1+1*(-1)+(-1)*(-1)+(-1)*1 = 1-1+1-1 = 0,\\
\mathbf{C}_2^t\mathbf{C}_3 &= 1*1+(-1)*(-1)+1*(-1)+(-1)*1 = 1+1-1-1 = 0.
\end{aligned}$$

Hence the three contrasts are orthogonal to each other.

(c) The SAS output for contrast sums of squares and contrasts testing is as follows.

| Contrast | DF | Contrast SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| C1 | 1 | 864.000000 | 864.000000 | 3.00 | 0.0989 |
| C2 | 1 | 5162.666667 | 5162.666667 | 17.90 | 0.0004 |
| C3 | 1 | 0.166667 | 0.166667 | 0.00 | 0.9811 |

From the table, Contrast $\mathbf{C}_1$ is close to significant but not significant ($p$-value = 0.0989), this tells us that the average effect of hormone I and the average effect of hormone II on are not different from each other; Contrast $\mathbf{C}_2$ is very significant ($p$-value = 0.0004), this shows that the average effect for high levels of hormones and the average effect for low levels of hormones are quite different; Contrast $\mathbf{C}_3$ is not significant at all ($p$-value = 0.9811), so the difference between the high-level and low-level of hormone I is the same as that between the high-level and low-level of hormone II. $\blacksquare$

## Problem 2

(a) The output ANOVA table is as follows (the extra output under the ANOVA table will be used in (b)).

| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 3 | 20.25200000 | 6.75066667 | 17.55 | <.0001 |

| | | | |
|---|---|---|---|
| Error | 16 | 6.15600000 | 0.38475000 |
| Corrected Total | 19 | 26.40800000 | |

| R-Square | Coeff Var | Root MSE | den Mean |
|---|---|---|---|
| 0.766889 | 2.866369 | 0.620282 | 21.64000 |

Since the $p$-value is very small ($< 0.0001$), I conclude that different firing temperatures have different effects on the density of brick.

(b) We have 4 different treatments here. So according to Table X or Table IX in the textbook, the orthogonal contrasts we should use are $\mathbf{C}_1 = (-3, -1, 1, 3)^t$, $\mathbf{C}_2 = (1, -1, -1, 1)^t$, and $\mathbf{C}_3 = (-1, 3, -3, 1)^t$, which correspond to the linear, quadratic and cubic effects (trends), respectively. Contrast statements in SAS result in the following output:

| Contrast | DF | Contrast SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| C1 | 1 | 3.16840000 | 3.16840000 | 8.23 | 0.0111 |
| C2 | 1 | 16.20000000 | 16.20000000 | 42.11 | <.0001 |
| C3 | 1 | 0.88360000 | 0.88360000 | 2.30 | 0.1492 |

The first two contrasts are significant but the third one is not. Hence, I conclude that the linear and quadratic effects are significant but the cubic effect is not. So my final orthogonal polynomial model contains only the linear and quadratic effects, and has the form

$$f(t) = \beta_0 + \beta_1 P_1(t) + \beta_2 P_2(t),$$

where $P_i(t)$ are the first two orthogonal polynomials.

It can be calculated that $\bar{y}_{\cdot\cdot} = 21.64$. Estimate statements in SAS give

| Parameter | Estimate | Standard Error | t Value | Pr > \|t\| |
|---|---|---|---|---|
| c1 | 3.56000000 | 1.24056439 | 2.87 | 0.0111 |
| c2 | 3.60000000 | 0.55479726 | 6.49 | <.0001 |
| c3 | -1.88000000 | 1.24056439 | -1.52 | 0.1492 |

From Table X or table IX, we have $\mathcal{D}_1 = 20$ and $\mathcal{D}_2 = 4$. Hence the estimates for $\beta_0$, $\beta_1$ and $\beta_2$ are

$$\hat{\beta}_0 = 21.64$$

$$\hat{\beta}_1 = \frac{C1}{\mathcal{D}_1} = \frac{3.56}{20} = 0.18$$

$$\hat{\beta}_2 = \frac{C2}{\mathcal{D}_2} = \frac{3.60}{4} = 0.90$$

Hence the fitted orthogonal polynomial model is

$$f(t) = 21.64 + 0.18P_1(t) + 0.90P_2(t).$$

(c) The median $m$ of the temperature levels is $\frac{125+150}{2} = 137.5$ and the difference $\delta$ between consecutive levels is 25. Also, $\lambda_1 = 2$ and $\lambda_2 = 1$ from Table X to Table IX. So

$$\begin{aligned} P_1(t) &= \lambda_1 \left(\frac{t-m}{\delta}\right) = \frac{2}{25}(t - 137.5), \\ P_2(t) &= \lambda_2 \left[\left(\frac{t-m}{\delta}\right)^2 - \frac{a^2-1}{12}\right] = \left[\left(\frac{t-137.5}{25}\right)^2 - \frac{5}{4}\right] \end{aligned}$$

After plugging these into the model in part (b) and doing some algebraic manipulations, I get

$$f(t) = a_1(t - 132.5)^2 + a_2,$$

where $a_1 > 0, a_2 > 0$ are two constants. Hence the temperature that produces the lowest density is 132.5. $\blacksquare$