Lecture 8: Balanced Incomplete Block Design

Montgomery Section 4.4

Catalyst Experiment

Four catalysts are being investigated in an experiment. The experimental procedure consists of selecting a batch raw material, loading the pilot plant, applying each catalyst in a separate run and observing the reaction time. The batches of raw material are considered as blocks, however each batch is only large enough to permit three catalysts to be run.

	Block(raw material)				
Catalyst	1	2	3	4	$y_{i.}$
1	73	74	-	71	218
2	-	75	67	72	214
3	73	75	68	-	216
4	75	-	72	75	222
$y_{.j}$	221	224	207	218	870=$y_{..}$

Balanced Incomplete Block Design (BIBD)

Example 1.

	block					
treatment	1	2	3			
A	A	-	A	1	0	1
B	B	B	-	1	1	0
C	-	C	C	0	1	1

$a = 3$, $b = 3$, $k = 2$, $r = 2$, $\lambda = 1$

Incidence Matrix: $\mathcal{N} = (n_{ij})_{a \times b}$ where $n_{ij} = 1$, if treatment i is run in block j;

=0 otherwise.

Example 2.

	block											
treatment	1	2	3	4	5	6						
A	A	A	A	-	-	-	1	1	1	0	0	0
B	B	-	-	B	B	-	1	0	0	1	1	0
C	-	C	-	C	-	C	0	1	0	1	0	1
D	-	-	D	-	D	D	0	0	1	0	1	1

$a = 4, b = 6, k = 2, r = 3, \lambda = 1, \mathcal{N} = (n_{ij})_{4\times 6}$

BIBD: Design Properties

- there are a treatments and b blocks.
- each block contains k (different) treatments.
- each treatment appears in r blocks.
- each pair of treatments appears together in λ blocks.

a, b, k, r, and λ are not independent

- $N = ar = bk$, where N is the total number of runs;

- $\lambda(a-1) = r(k-1)$:

  1. for any fixed treatment i_0

  2. two different ways to count the total number of pairs including treatment i_0 in the experiment.

  I. $a-1$ possible pairs, each appears in λ blocks, so $\lambda(a-1)$;

  II. treatment i_0 appears in r blocks. Within each block, there are $k-1$ pairs including i_0, so $r(k-1)$

- $b \geq a$ (a brainteaser for math/stat students).

- Nonorthogonal design

Extensive list of BIBDs can be found in Fisher and Yates (1963) and Cochran and Cox (1957).

BIBD: Statistical Model

- Statistical Model

$$y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}$$

- additive model (without interaction)
- Not all y_{ij} exist because of incompleteness
- Usual treatment and block restrictions : $\sum \tau_i = 0; \qquad \sum \beta_j = 0$
- Nonorthogonality of treatments and blocks

Use Type III Sums of Squares and lsmeans

Model Estimates

- Least squares estimates for μ, etc.

$$\hat{\mu} = \frac{y_{..}}{N}; \qquad \hat{\tau}_i = \frac{kQ_i}{\lambda a}; \qquad \hat{\beta}_j = \frac{rQ'_j}{\lambda b}$$

where

$$Q_i = y_{i.} - \frac{1}{k}\sum n_{ij} y_{.j}; \qquad Q'_j = y_{.j} - \frac{1}{r}\sum n_{ij} y_{i.}$$

$$\begin{aligned}
\mathrm{Var}(Q_i) &= \mathrm{Var}\left(y_{i.}\right) + \mathrm{Var}\left(\frac{1}{k}\sum n_{ij} y_{.j}\right) - 2\mathrm{Cov}\left(y_{i.}, \frac{1}{k}\sum n_{ij} y_{ij}\right) \\
&= r\sigma^2 + \frac{r}{k^2} k\sigma^2 - \frac{2}{k} r\sigma^2 \\
&= \frac{(k-1)r}{k}\sigma^2
\end{aligned}$$

- $\mathrm{Var}\left(\hat{\tau}_i\right) = \left(\frac{k}{\lambda a}\right)^2 \mathrm{Var}(Q_i) = \left(\frac{k}{\lambda a}\right)^2 \frac{(k-1)r}{k}\sigma^2 = \frac{k(a-1)}{\lambda a^2}\sigma^2; \;\; \mathrm{S.E.}_{\hat{\tau}_i} = ?$
- $\mathrm{Var}\left(\hat{\tau}_i - \hat{\tau}_j\right) = \frac{2k\sigma^2}{\lambda a}; \;\; \mathrm{S.E.}_{\hat{\tau}_i - \hat{\tau}_j} = ?$

Analysis of Variance Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Blocks	SS_{Block}	$b-1$	MS_{Block}	
Treatment	$\text{SS}_{\text{Treatment}}$	$a-1$	$\text{MS}_{\text{Treatment}}$	F_0
Error	SS_{E}	$N-a-b+1$	MS_{E}	
Total	SS_{T}	$N-1$		

- $\text{SS}_{\text{T}} = \sum\sum y_{ij}^2 - y_{..}^2/N$
- $\text{SS}_{\text{Block}} = \frac{1}{k}\sum y_{.j}^2 - y_{..}^2/N$

- $\mathrm{SS}_{\mathrm{Treatments}}$ needs adjustment for incompleteness

$$Q_i = y_{i.} - \frac{1}{k}\sum_{j=1}^{b} n_{ij} y_{.j} \quad \text{where} \quad n_{ij} = \begin{cases} 1 & \text{if trt } i \text{ in blk } j \\ 0 & \text{otherwise} \end{cases}$$

  - trt i's **total** minus trt i's block averages
  - $\sum Q_i = 0$

$$\mathrm{SS}_{\mathrm{Treatment(adjusted)}} = k \sum Q_i^2/\lambda a = \frac{\lambda a}{k} \sum \hat{\tau}_i^2$$

- SS_{E} by subtraction
- If $F_0 > F_{\alpha, a-1, N-a-b+1}$ then reject H_0

Mean Tests and Contrasts

- Must compute adjusted means (lsmeans)
- Adjusted mean is $\widehat{\mu} + \widehat{\tau}_i$
- Standard error of adjusted mean is $\sqrt{\mathrm{MS_E}(\frac{k(a-1)}{\lambda a^2} + \frac{1}{N})}$
- Contrasts based on adjusted treatment totals

  For a contrast: $\sum c_i \mu_i$

  Its estimate: $\sum c_i \hat{\tau}_i = \frac{k}{\lambda a} \sum c_i Q_i$

  Contrast sum of squares:

$$SS_C = \frac{k(\sum_{i=1}^{a} c_i Q_i)^2}{\lambda a \sum_{i=1}^{a} c_i^2}$$

Pairwise Comparison

- Pairwise comparison $\tau_i - \tau_j$:

1. Bonferroni:

$$CD = t_{\alpha/2m,ar-a-b+1}\sqrt{\mathrm{MS_E}\frac{2k}{\lambda a}}.$$

2. Tukey:

$$CD = \frac{q_\alpha(a, ar-a-b+1)}{\sqrt{2}}\sqrt{\mathrm{MS_E}\frac{2k}{\lambda a}}$$

SAS Code and output

```
options nocenter ps=60 ls=75;
data example;
 input trt block resp @@;
 datalines;
1 1 73 1 2 74 1 4 71 2 2 75 2 3 67 2 4 72
3 1 73 3 2 75 3 3 68 4 1 75 4 3 72 4 4 75
;

proc glm;
class block trt;
model resp = block trt;
lsmeans trt / tdiff pdiff adjust=bon stderr;
lsmeans trt / pdiff adjust=tukey;
contrast ’a’ trt 1 -1 0 0;
estimate ’b’ trt 0 0 1 -1;
run;
```

SAS output

```
Dependent Variable: resp
                          Sum of
Source             DF    Squares   Mean Square  F Value  Pr > F
Model               6  77.75000000  12.95833333   19.94  0.0024
Error               5   3.25000000   0.65000000
Corrected Total    11  81.00000000

Source             DF   Type I SS   Mean Square  F Value  Pr > F
block               3  55.00000000  18.33333333   28.21  0.0015
trt                 3  22.75000000   7.58333333   11.67  0.0107


Source             DF  Type III SS  Mean Square  F Value  Pr > F
block              3  66.08333333  22.02777778   33.89  0.0010
trt                3  22.75000000   7.58333333   11.67  0.0107

------------------------------------------------------------------
Least Squares Means
Adjustment for Multiple Comparisons: Bonferroni
```

trt	resp LSMEAN	Standard Error	Pr > |t|	LSMEAN Number
1	71.3750000	0.4868051	<.0001	1
2	71.6250000	0.4868051	<.0001	2
3	72.0000000	0.4868051	<.0001	3
4	75.0000000	0.4868051	<.0001	4

Bonferroni Method:

i/j	1	2	3	4
1		-0.35806	-0.89514	-5.19183
		1.0000	1.0000	0.0209
2	0.358057		-0.53709	-4.83378
	1.0000		1.0000	0.0284
3	0.895144	0.537086		-4.29669
	1.0000	1.0000		0.0464
4	5.191833	4.833775	4.296689	
	0.0209	0.0284	0.0464	

Tukey’s Method:

i/j	1	2	3	4
1		0.9825	0.8085	0.0130
2	0.9825		0.9462	0.0175
3	0.8085	0.9462		0.0281
4	0.0130	0.0175	0.0281	

```
Dependent Variable: resp

Contrast   DF   Contrast SS   Mean Square  F Value  Pr > F
c1          1   0.08333333    0.08333333     0.13   0.7349

                                Standard
Parameter   Estimate           Error         t Value   Pr > |t|
b         -3.00000000     0.69821200       -4.30     0.0077
```

Other Incomplete Designs

- Youden Square
- Partially Balanced Incomplete Block Design
- Cyclic Designs
- Square, Cubic, and Rectangular Lattices