# Lecture 3. Experiments with a Single Factor: ANOVA

Montgomery 3-1 through 3-3

## Tensile Strength Experiment

Investigate the tensile strength of a new synthetic fiber. The factor is the weight percent of cotton used in the blend of the materials for the fiber and it has five levels.

| percent | tensile strength | | | | | | |
|---|---|---|---|---|---|---|---|
| of cotton | 1 | 2 | 3 | 4 | 5 | total | average |
| 15 | 7 | 7 | 11 | 15 | 9 | 49 | 9.8 |
| 20 | 12 | 17 | 12 | 18 | 18 | 77 | 15.4 |
| 25 | 14 | 18 | 18 | 19 | 19 | 88 | 17.6 |
| 30 | 19 | 25 | 22 | 19 | 23 | 108 | 21.6 |
| 35 | 7 | 10 | 11 | 15 | 11 | 54 | 10.8 |

## Data Layout for Single-Factor Experiments

| treatment | observations | | | | totals | averages |
|---|---|---|---|---|---|---|
| 1 | $y_{11}$ | $y_{12}$ | $\cdots$ | $y_{1n}$ | $y_{1.}$ | $\bar{y}_{1.}$ |
| 2 | $y_{21}$ | $y_{22}$ | $\cdots$ | $y_{2n}$ | $y_{2.}$ | $\bar{y}_{2.}$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\ddots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| a | $y_{a1}$ | $y_{a2}$ | $\cdots$ | $y_{an}$ | $y_{a.}$ | $\bar{y}_{a.}$ |

## Analysis of Variance

- Interested in comparing several treatments.
- Could do numerous two-sample t-tests but this approach does not test equality of all treatments simultaneously.
- ANOVA provides a method of joint inference.
- Statistical Model:

$$y_{ij} = \mu + \tau_i + \epsilon_{ij} \quad \begin{cases} i = 1, 2 \ldots a \\ j = 1, 2, \ldots n_i \end{cases}$$

  $\mu$ - grand mean; $\tau_i$ - $i$th treatment effect; $\epsilon_{ij} \sim N(0, \sigma^2)$ - error

  Constraint: $\sum_{i=1}^{a} \tau_i = 0$.

- Basic Hypotheses:

  $H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = 0$ vs $H_1 : \tau_i \neq 0$ for at least one $i$

## Derive Estimates: Partitioning $y_{ij}$

- Notation
  - $y_{i.} = \sum_{j=1}^{n_i} y_{ij} \rightarrow \overline{y}_{i.} = y_{i.}/n_i$ (treatment sample mean, or row mean)
  - $y_{..} = \sum\sum y_{ij} \rightarrow \overline{y}_{..} = y_{..}/N$ (grand sample mean)
- Decomposition of $y_{ij}$: $\quad y_{ij} = \overline{y}_{..} + (\overline{y}_{i.} - \overline{y}_{..}) + (y_{ij} - \overline{y}_{i.})$
- Estimates for parameters:

$$\hat{\mu} = \overline{y}_{..}$$

$$\hat{\tau}_i = (\overline{y}_{i.} - \overline{y}_{..})$$

$$\hat{\epsilon}_{ij} = y_{ij} - \overline{y}_{i.} \qquad (\text{ residual })$$

  So $y_{ij} = \hat{\mu} + \hat{\tau}_i + \hat{\epsilon}_{ij}$.

  - It can be verified that

    $\sum_{i=1}^{a} n_i \hat{\tau}_i = 0; \qquad \sum_{j=1}^{n_i} \hat{\epsilon}_{ij} = 0$ for all $i$.

## Test Basic Hypotheses: Partitioning the Sum of Squares

- Recall $y_{ij} - \overline{y}_{..} = \hat{\tau}_i + \hat{\epsilon}_{ij} = (\overline{y}_{i.} - \overline{y}_{..}) + (y_{ij} - \overline{y}_{i.})$
- Can show

$$\begin{aligned} \sum_i \sum_j (y_{ij} - \overline{y}_{..})^2 = & \quad \sum_i n_i (\overline{y}_{i.} - \overline{y}_{..})^2 \quad + \sum_i \sum_j (y_{ij} - \overline{y}_{i.})^2 \\ = & \quad \sum_i n_i \hat{\tau}_i^2 \quad + \sum_i \sum_j \hat{\epsilon}_{ij}^2. \\ \text{Total } \mathrm{SS} = & \quad \text{Treatment } \mathrm{SS} \quad + \text{Error } \mathrm{SS} \end{aligned}$$

$\mathrm{SS_T}$ = $\mathrm{SS_{Treatments}}$ + $\mathrm{SS_E}$

- Derive test statistics for testing $H_0 : \tau_1 = \cdots = \tau_a = 0$:
  look at $\mathrm{SS_{Treatments}}$= $\sum n_i \hat{\tau}_i^2 = \sum n_i (\overline{y}_{i.} - \overline{y}_{..})^2$

  Small if $\left|\hat{\tau}_i\right|'s$ are small

  If large then reject $H_0$, but how large is large?

  Standardize to account for inherent variability

## Test Statistic

$$F_0 = \frac{\mathrm{SS}_{\mathrm{Treatments}}/(a-1)}{\mathrm{SS}_{\mathrm{E}}/(N-a)} = \frac{\mathrm{MS}_{\mathrm{Treatments}}}{\mathrm{MS}_{\mathrm{E}}}$$

- Mean squares:

  $\mathrm{MS}_{\mathrm{Treatments}} = \sum_{i=1}^{a} n_i\hat{\tau}_i/(a-1)$ $\qquad$ $\mathrm{MS}_{\mathrm{E}} = \sum_i \sum_j \hat{\epsilon}_{ij}^2/(N-a)$

- Can show that the expected values are

  E($\mathrm{MS}_{\mathrm{E}}$)=$\sigma^2$

  E($\mathrm{MS}_{\mathrm{Treatment}}$) = $\sigma^2 + \sum n_i\tau_i^2/(a-1)$

- Hence, $\mathrm{MS}_{\mathrm{E}}$ is always a good (unbiased) estimator for $\sigma^2$, that is, $\hat{\sigma}^2$ =$\mathrm{MS}_{\mathrm{E}}$.

  Under $H_0$, $\mathrm{MS}_{\mathrm{Treatment}}$ is also a good (unbiased) estimator for $\sigma^2$.

  Under $H_0$, $F_0$ is expected close to 1. Large $F_0$ implies deviation from $H_0$.

- What is the sampling distribution of $F_0$ under $H_0$?

## Sampling Distribution of $F_0$ under $H_0$

- Based on model:

  $y_{ij} \sim \mathrm{N}(\mu + \tau_i, \sigma^2)$ implies $\overline{y}_{i.} \sim \mathrm{N}(\mu + \tau_i, \sigma^2/n_i)$

  implies $\sum_j (y_{ij} - \overline{y}_{i.})^2/\sigma^2 \sim \chi^2_{n_i - 1}$

  **Therefore**:

  $\mathrm{SS_E}/\sigma^2 = \sum_i \sum_j (y_{ij} - \overline{y}_{i.})^2/\sigma^2 \sim \chi^2_{N-a}$

- Under $H_0$:

  $\mathrm{SS_{Treatment}}/\sigma^2 = \sum_i n_i(\overline{y}_{i.} - \overline{y}_{..})^2/\sigma^2 \sim \chi^2_{a-1}$

- $\mathrm{SS_E}$ and $\mathrm{SS_{Treatment}}$ are independent. So

$$F_0 = \frac{\mathrm{SS_{Treatment}}/\sigma^2(a-1)}{\mathrm{SS_E}/\sigma^2(N-a)} = \frac{\chi^2_{a-1}/(a-1)}{\chi^2_{N-a}/(N-a)} \sim F_{a-1,N-a}$$

  $F$-distribution with numerator df $a-1$ and denominator df $N-a$.

## Analysis of Variance (ANOVA) Table

| Source of Variation | Sum of Squares | Degrees of Freedom | Mean Square | $F_0$ |
|---|---|---|---|---|
| Between | $\text{SS}_{\text{Treatment}}$ | $a-1$ | $\text{MS}_{\text{Treatment}}$ | $F_0$ |
| Within | $\text{SS}_{\text{E}}$ | $N-a$ | $\text{MS}_{\text{E}}$ | |
| Total | $\text{SS}_{\text{T}}$ | $N-1$ | | |

If balanced:

$$\text{SS}_{\text{T}} = \sum\sum y_{ij}^2 - y_{..}^2/N; \qquad \text{SS}_{\text{Treatment}} = \frac{1}{n}\sum y_{i.}^2 - y_{..}^2/N$$

$\text{SS}_{\text{E}}$=$\text{SS}_{\text{T}}$ - $\text{SS}_{\text{Treatment}}$

If unbalanced:

$$\text{SS}_{\text{T}} = \sum\sum y_{ij}^2 - y_{..}^2/N; \qquad \text{SS}_{\text{Treatment}} = \sum \frac{y_{i.}^2}{n_i} - y_{..}^2/N$$

$\text{SS}_{\text{E}}$=$\text{SS}_{\text{T}}$ - $\text{SS}_{\text{Treatment}}$

- **Decision Rule**: If $F_0 > F_{\alpha,a-1,N-a}$ then reject $H_0$

## Connection between F-Test and Two-Sample t-Test

- a=2
- Consider the square of the t-test statistic

$$t_0^2 = \frac{(\bar{y}_{1.} - \bar{y}_{2.})^2}{(s_{pool}\sqrt{1/n_1 + 1/n_2})^2} = \frac{\frac{n_1 n_2}{n_1+n_2}(\bar{y}_{1.} - \bar{y}_{2.})^2}{s_{pool}^2}$$

$$= \frac{\frac{n_1 n_2}{n_1+n_2}[(\bar{y}_{1.} - \bar{y}_{..}) - (\bar{y}_{2.} - \bar{y}_{..})]^2}{s_{pool}^2}$$

$$= \frac{\frac{n_1 n_2}{n_1+n_2}(\bar{y}_{1.} - \bar{y}_{..})^2 + \frac{n_1 n_2}{n_1+n_2}(\bar{y}_{2.} - \bar{y}_{..})^2 - \frac{2n_1 n_2}{n_1+n_2}(\bar{y}_{1.} - \bar{y}_{..})(\bar{y}_{2.} - \bar{y}_{..})}{s_{pool}^2}$$

Consider

$$\bar{y}_{..} = \frac{1}{n_1+n_2}(\sum_{j=1}^{n_1} y_{1j} + \sum_{j=1}^{n_2} y_{2j}) = \frac{n_1}{n_1+n_2}\bar{y}_{1.} + \frac{n_2}{n_1+n_2}\bar{y}_{2.}$$

One has

$$t_0^2 = \frac{[n_1(\bar{y}_{1.} - \bar{y}_{..})^2 + n_2(\bar{y}_{2.} - \bar{y}_{..})^2]/(2-1)}{[\sum_{j=1}^{n_1}(y_{1j} - \bar{y}_{1.})^2 + \sum_{j=1}^{n_2}(y_{2j} - \bar{y}_{2.})^2]/(n_1+n_2-2)}$$

$$= \frac{\mathrm{MS}_{\mathrm{Treatment}}}{\mathrm{MS}_{\mathrm{E}}}$$

- When $a = 2$, $t_0^2 = F_0$
- When $a = 2$, $F$-test and two-sample two-sided test are equivalent.

**Example**

Twelve lambs are randomly assigned to three different diets. The weight gain (in two weeks) is recorded. Is there a difference between the diets?

| | | | | | |
|---|---|---|---|---|---|
| Diet 1 | 8 | 16 | 9 | | |
| Diet 2 | 9 | 16 | 21 | 11 | 18 |
| Diet 3 | 15 | 10 | 17 | 6 | |

- $N = 12$, $\sum\sum y_{ij} = 156$ and $\bar{y}_{..} = 156/12 = 13$.
- $n_1 = 3$, $y_{1.} = 33$, $\bar{y}_{1.} = 11$; $n_2 = 5$, $y_{2.} = 75$, $\bar{y}_{2.} = 15$; $n_3 = 4$, $y_{3.} = 48$ and $\bar{y}_{3.} = 12$.
- $\hat{\tau}_1 = \bar{y}_{1.} - \bar{y}_{..} = 11 - 13 = -2$; Similarly, $\hat{\tau}_2 = 15 - 13 = 2$ and $\hat{\tau}_3 = 12 - 13 = -1$.
- $\mathrm{SS_T} = \sum_i \sum_j (y_{ij} - \bar{y}_{..})^2 = 246$.
- $\mathrm{SS_{Treatment}} = 3 * (-2)^2 + 5 * (2)^2 + 4 * (-1)^2 = 36$.
- $\mathrm{SS_E} = 246 - 36 = 210$

- $\mathrm{MS}_E = \hat{\sigma}^2 = 210/(12-3) = 23.33$
- $F_0 = (36/2)/(210/9) = 0.77$; P-value $> 0.25$
- Fail to reject $H_0 : \tau_1 = \tau_2 = \tau_3 = 0$.

## Using SAS (lambs.sas)

```
option nocenter ps=65 ls=80;

data lambs;
 input diet wtgain;
 datalines;
  1  8
  1 16
  1  9
  2  9
  2 16
  2 21
  2 11
  2 18
  3 15
  3 10
  3 17
  3  6
;
```

```
symbol1 bwidth=5 i=box; axis1 offset=(5);
proc gplot; plot wtgain*diet / frame haxis=axis1;

proc glm;
 class diet;
 model wtgain=diet;
 output out=diag r=res p=pred;

proc gplot; plot res*diet /frame haxis=axis1;

proc sort; by pred;
symbol1 v=circle i=sm50;
proc gplot; plot res*pred / haxis=axis1;
run;
```

## SAS Output

The GLM Procedure
Dependent Variable: wtgain

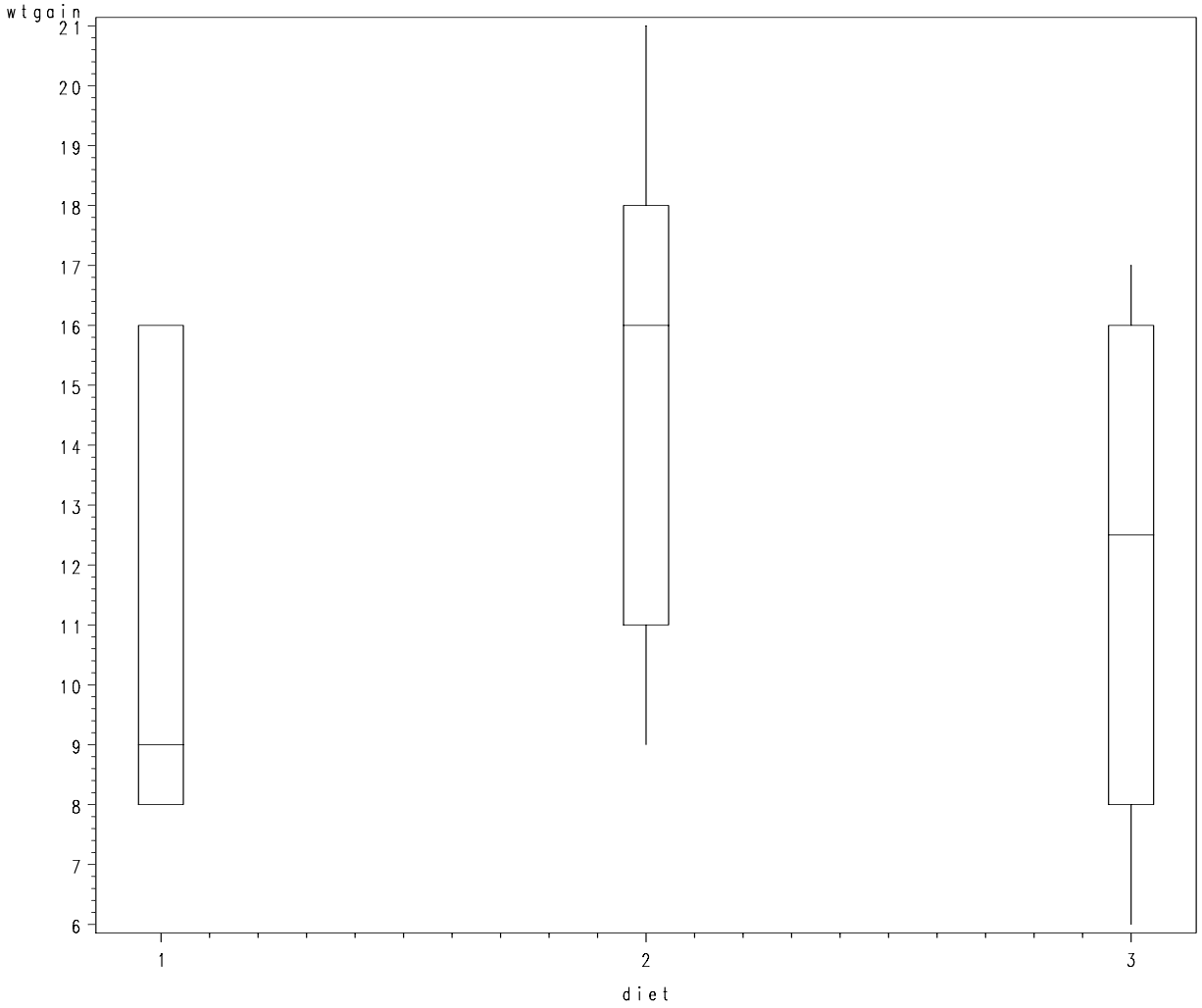
| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 2 | 36.0000000 | 18.0000000 | 0.77 | 0.4907 |
| Error | 9 | 210.0000000 | 23.3333333 | | |
| Corrected Total | 11 | 246.0000000 | | | |

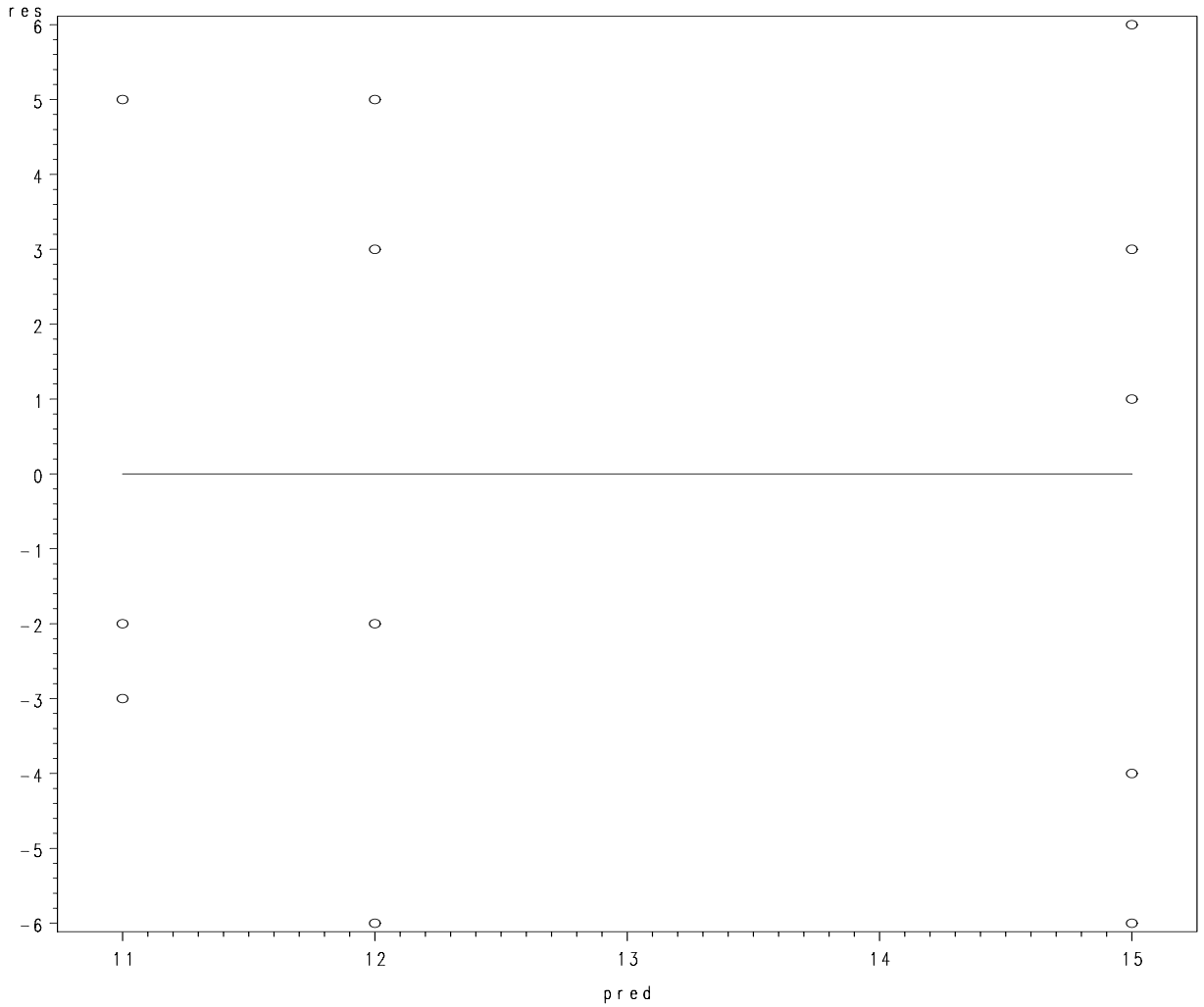
| R-Square | Coeff Var | Root MSE | wtgain Mean |
|---|---|---|---|
| 0.146341 | 37.15738 | 4.830459 | 13.00000 |

| Source | DF | Type I SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| diet | 2 | 36.00000000 | 18.00000000 | 0.77 | 0.4907 |

| Source | DF | Type III SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| diet | 2 | 36.00000000 | 18.00000000 | 0.77 | 0.4907 |

## Side-by-Side Boxplot






## Residual Plot

## Tensile Strength Experiment

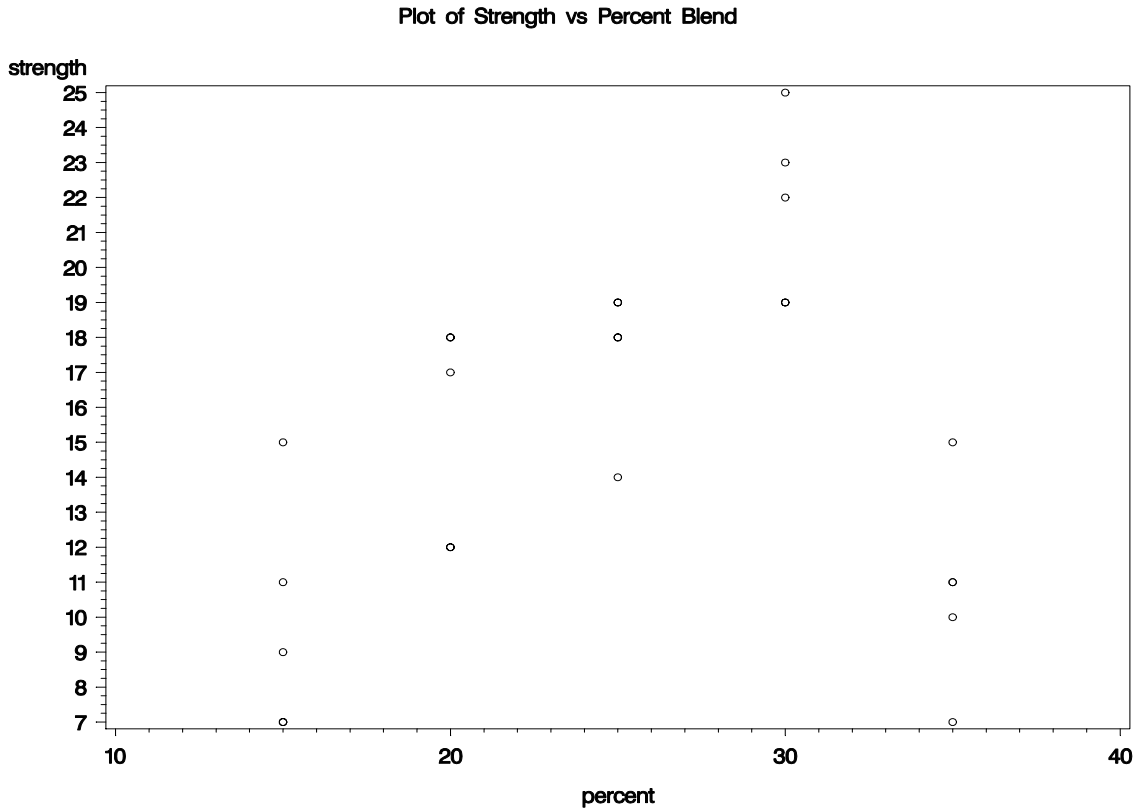
```
options ls=80 ps=60 nocenter;
goptions device=win target=winprtm rotate=landscape ftext=swiss
   hsize=8.0in vsize=6.0in htext=1.5 htitle=1.5 hpos=60 vpos=60
   horigin=0.5in vorigin=0.5in;
data one;
 infile 'c:\saswork\data\tensile.dat';
 input percent strength time;

title1 'Tensile Strength Example';
proc print data=one; run;

symbol1 v=circle i=none;
title1 'Plot of Strength vs Percent Blend';
proc gplot data=one;  plot strength*percent/frame; run;

proc boxplot;
```

```
 plot strength*percent/boxstyle=skeletal pctldef=4;

proc glm;
 class percent; model strength=percent;
 output out=oneres p=pred r=res; run;

proc sort; by pred;
symbol1 v=circle i=sm50; title1 'Residual Plot';
proc gplot;  plot res*pred/frame; run;

proc univariate data=oneres normal;
 var res;  qqplot res / normal (L=1 mu=est sigma=est);
 histogram res / normal; run;

symbol1 v=circle i=none;
title1 'Plot of residuals vs time';
proc gplot; plot res*time / vref=0 vaxis=-6 to 6 by 1;
run;
```

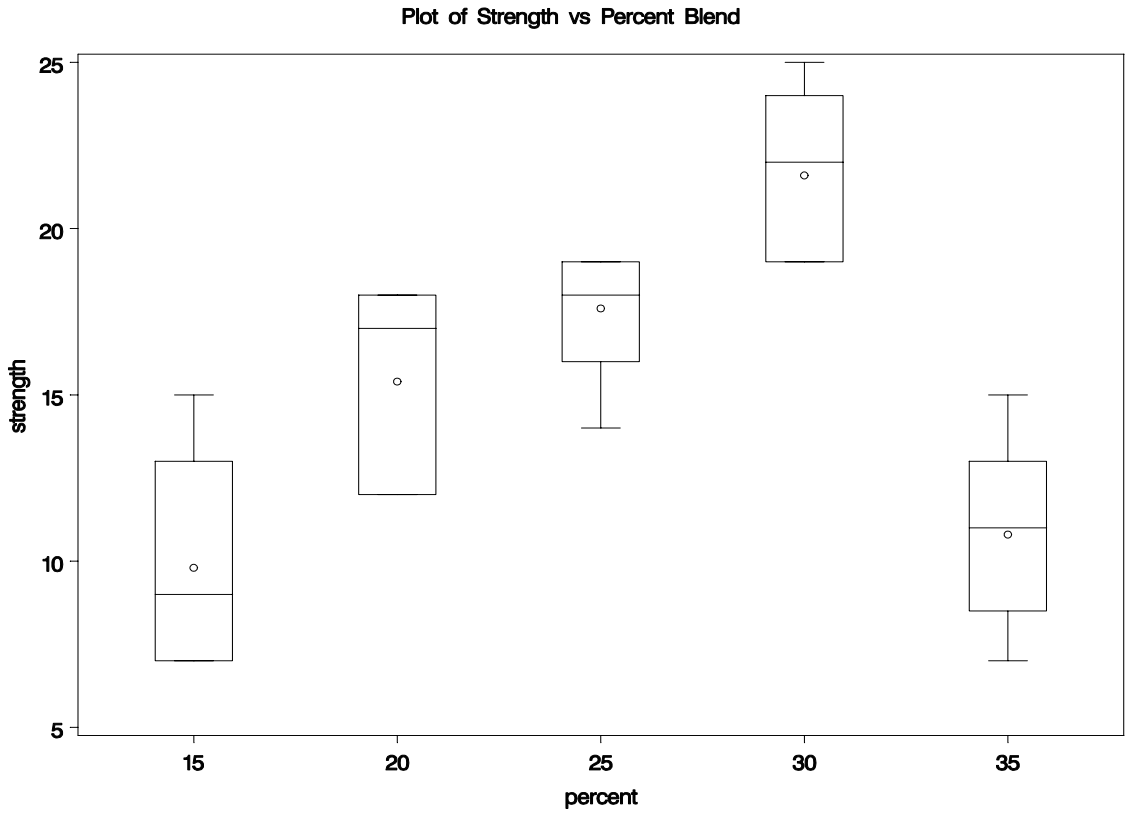
Tensile Strength Example

| Obs | percent | strength | time |
|---|---|---|---|
| 1 | 15 | 7 | 15 |
| 2 | 15 | 7 | 19 |
| 3 | 15 | 15 | 25 |
| 4 | 15 | 11 | 12 |
| 5 | 15 | 9 | 6 |
| 6 | 20 | 12 | 8 |
| 7 | 20 | 17 | 14 |
| 8 | 20 | 12 | 1 |
| 9 | 20 | 18 | 11 |
| : | : | : | : |
| 16 | 30 | 19 | 22 |
| 17 | 30 | 25 | 5 |
| 18 | 30 | 22 | 2 |
| 19 | 30 | 19 | 24 |
| 20 | 30 | 23 | 10 |
| 21 | 35 | 7 | 17 |
| 22 | 35 | 10 | 21 |
| 23 | 35 | 11 | 4 |
| 24 | 35 | 15 | 16 |
| 25 | 35 | 11 | 23 |

# Scatter Plot



# Side-by-Side Plot



The GLM Procedure
Dependent Variable: strength

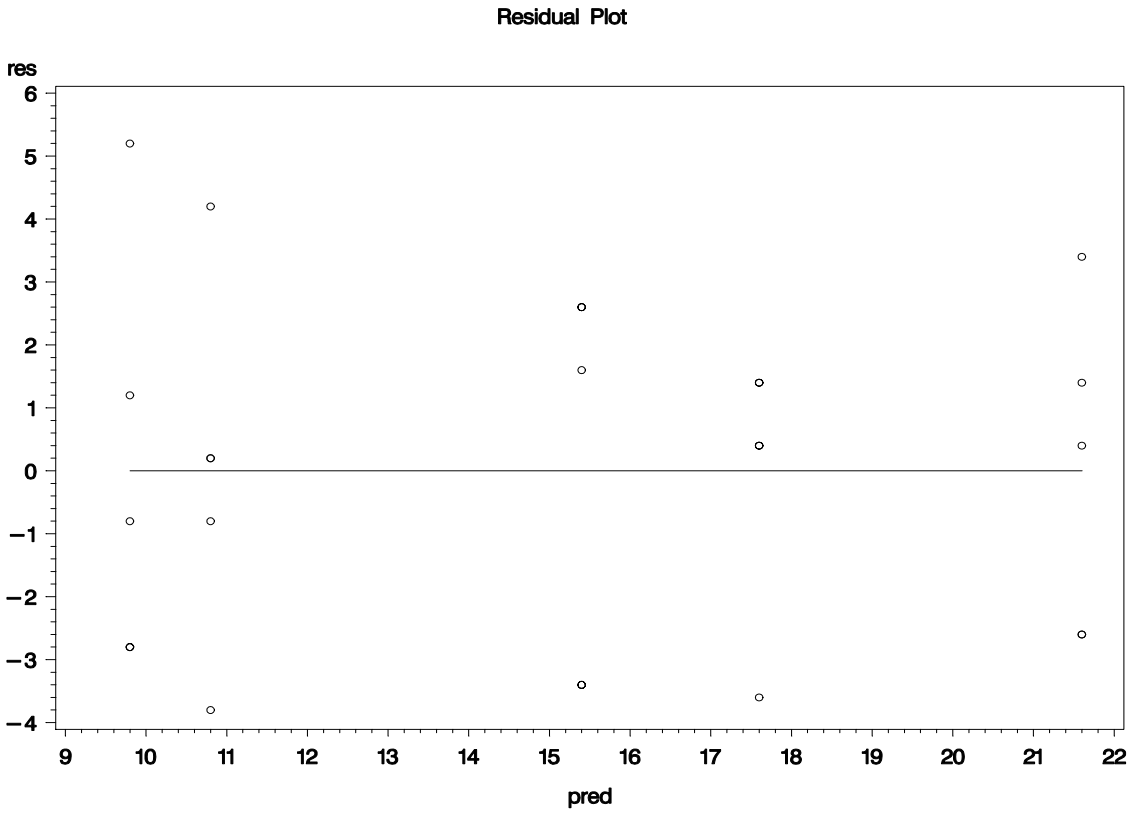
| Source | DF | Sum of Squares | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| Model | 4 | 475.7600000 | 118.9400000 | 14.76 | <.0001 |
| Error | 20 | 161.2000000 | 8.0600000 | | |
| Corrected Total | 24 | 636.9600000 | | | |

| R-Square | Coeff Var | Root MSE | strength Mean |
|---|---|---|---|
| 0.746923 | 18.87642 | 2.839014 | 15.04000 |

| Source | DF | Type I SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| percent | 4 | 475.7600000 | 118.9400000 | 14.76 | <.0001 |

| Source | DF | Type III SS | Mean Square | F Value | Pr > F |
|---|---|---|---|---|---|
| percent | 4 | 475.7600000 | 118.9400000 | 14.76 | <.0001 |

# Residual Plot



The UNIVARIATE Procedure
Variable: res

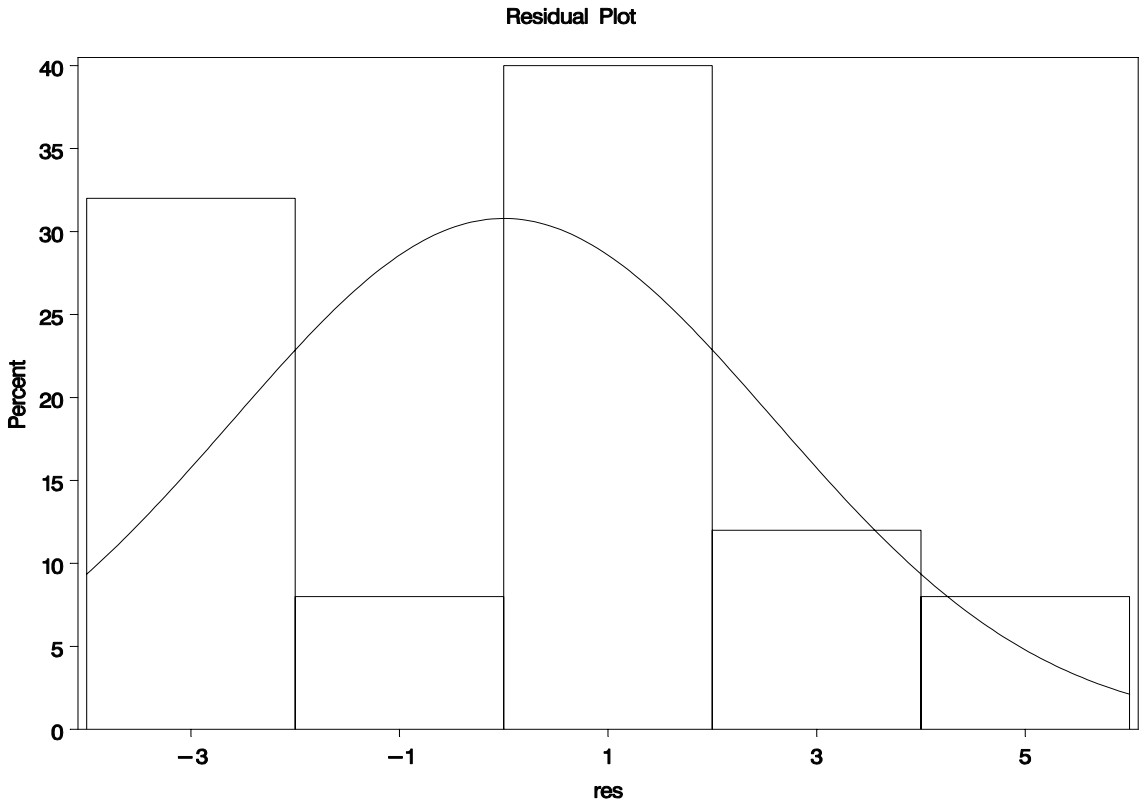
Moments

| | | | |
|---|---|---|---|
| N | 25 | Sum Weights | 25 |
| Mean | 0 | Sum Observations | 0 |
| Std Deviation | 2.59165327 | Variance | 6.71666667 |
| Skewness | 0.11239681 | Kurtosis | -0.8683604 |
| Uncorrected SS | 161.2 | Corrected SS | 161.2 |
| Coeff Variation | . | Std Error Mean | 0.51833065 |

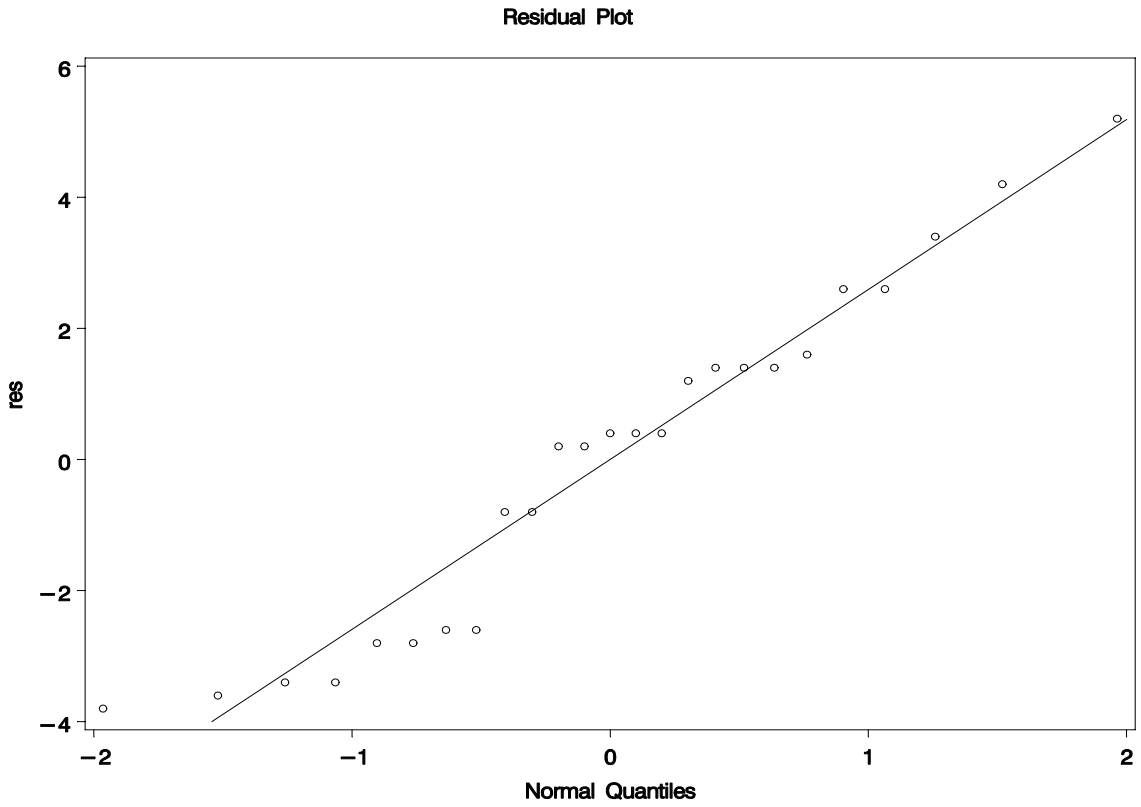
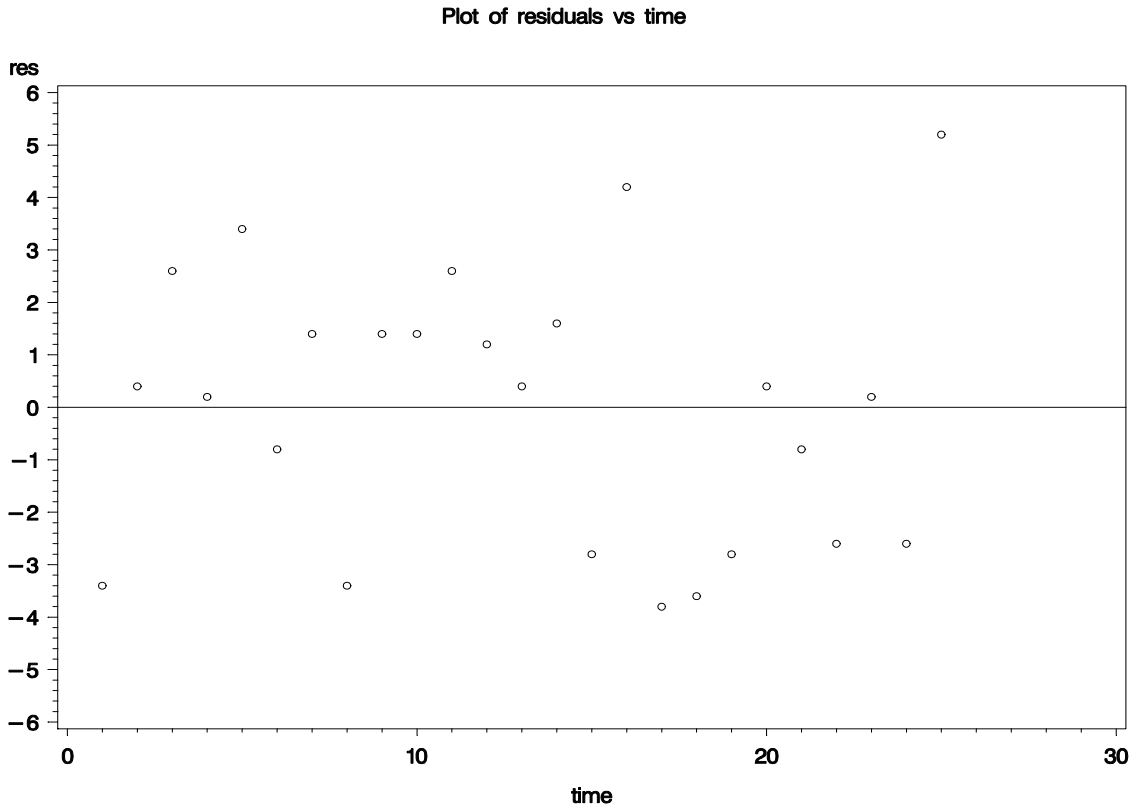
Tests for Normality

| Test | | --Statistic--- | -----p Value------ | |
|---|---|---|---|---|
| Shapiro-Wilk | W | 0.943868 | Pr < W | 0.1818 |
| Kolmogorov-Smirnov | D | 0.162123 | Pr > D | 0.0885 |
| Cramer-von Mises | W-Sq | 0.080455 | Pr > W-Sq | 0.2026 |
| Anderson-Darling | A-Sq | 0.518572 | Pr > A-Sq | 0.1775 |

# Histogram of Residuals



# QQ Plot



# Time Serial Plot