Ch5 Curved Exponential Families

$\mathcal{P} = \{ P_\eta : \eta \in \Xi \}$: a full rank s-parameter canonical exponential family.

T : complete sufficient statistic.

$\mathcal{P}_0 = \{ P_{\tilde{\eta}(\theta)} : \theta \in \Omega \}$.

$\tilde{\eta} : \Omega \to \Xi$ one-to-one and onto.

$\Omega \subset \mathbb{R}^r$ and $r < s$

① $\tilde{\eta}(\Omega) = \{ \tilde{\eta}(\theta) : \theta \in \Omega \}$

points η in the range of $\tilde{\eta}$ satisfy a nontrivial linear contranit.

$\mathcal{P}_0$: a q-parameter exponential family for $q < s$.

T : sufficient, will not be minimal sufficient.

② The points η in $\tilde{\eta}(\Omega)$ do not satisfy a linear contrant.

$\mathcal{P}_0$: a curved exponential family.

T : minimal suff. but may not be complete.

(Ex 3.12)

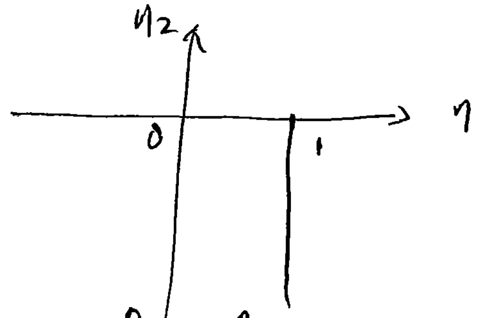
ex: $X_1 \cdots X_n \overset{iid}{\sim} N(\mu, \sigma^2)$

$\eta = \left(\frac{\mu}{\sigma^2}, -\frac{1}{2\sigma^2}\right)$: canonical parameter.

$T = \left(\sum_{i=1}^n X_i, \sum_{i=1}^n X_i^2\right)$: complete and suff.

if $\mu = \sigma^2 := \theta$. $N(\theta, \theta)$. $\theta > 0$

$\tilde{\eta}(\theta) = \left(1, -\frac{1}{2\theta}\right)$



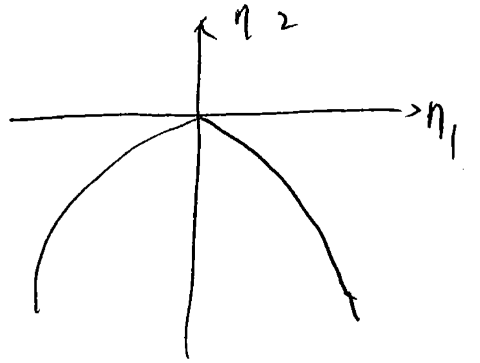
linear constraint: $\eta_1 = 1$.

$$\prod_{i=1}^n \frac{1}{\sqrt{2\pi\theta}} e^{-\frac{(x_i-\theta)^2}{2\theta}} = e^{-\sum_{i=1}^n x_i^2 \frac{1}{2\theta} + \sum x_i - \frac{n\theta}{2} - \sum_{i=1}^n \log \frac{1}{\sqrt{2\pi\theta}}}$$

rank-1 exponential family, full rank. $\sum_{i=1}^n X_i^2$ — complete suff.

$\sigma = |\mu| := |\theta|$ $N(\theta, \theta^2)$

$\tilde{\eta}(\theta) = \left(\frac{1}{\theta}, -\frac{1}{2\theta^2}\right)$



curved exponential family.

T — minimal suff.

$$E_\theta T_1^2 = (E_\theta T_1)^2 + Var_\theta(T_1) = n^2\theta^2 + n\theta^2$$

$$E_\theta T_2 = n E_\theta X_i^2 = 2n\theta^2.$$

$$E_\theta(2T_1^2 - (n+1)T_2) = 0. \qquad \theta \in \mathbb{R}.$$

interior

$$g(T) = 2T_1^2 - (n+1)T_2 \neq 0. \qquad T \text{ — not complete}$$

ex. $X_1 \cdots X_m \overset{iid}{\sim} N(\mu_X, \sigma_X^2)$

$Y_1 \cdots Y_n \overset{iid}{\sim} N(\mu_Y, \sigma_Y^2)$

$\theta = (\mu_X. \ \mu_Y \ \sigma_X^2. \ \sigma_Y^2)$. 4-parameter exponential family

$T = \left(\sum_{i=1}^m X_i \ \sum_{j=1}^n Y_j \ \sum_{i=1}^m X_i^2 \ \sum_{j=1}^n Y_j^2 \right)$ — suff.

$\eta = \left(\frac{\mu_X}{\sigma_X^2}. \ \frac{\mu_Y}{\sigma_Y^2}. \ -\frac{1}{2\sigma_X^2}. \ -\frac{1}{2\sigma_Y^2} \right)$ — canonical parameter

$(\bar{X}. \bar{Y}, S_X^2, S_Y^2)$.

If $\sigma_X^2 = \sigma_Y^2 = \sigma^2$. η satisfies the linear constraint $\underline{\eta_3 = \eta_4}$

3-parameter exponential family

$(T_1. \ T_2. \ T_3 + T_4)$ — complete and suff.

$\Downarrow$

$(\bar{X} \cdot \bar{Y}, \ S_p^2)$

$$S_p^2 = \frac{\sum_{i=1}^{m}(X_i-\bar{X})^2 + \sum_{i=1}^{n}(Y_i-\bar{Y})^2}{m+n-2} = \frac{(m-1)S_X^2+(n-1)S_Y^2}{n+m-2}.$$

pooled sample variance.

$$\frac{(m+n-2)S_p^2}{\sigma^2} \sim \chi^2_{n+m-2}.$$

If $\mu_X=\mu_Y$ $\rightarrow$ curved exponential family.

T or $(\bar{X}, \bar{Y}, S_X^2, S_Y^2)$ — minimal suff.
not complete.

$E_\theta(\bar{X}-\bar{Y})=0$

- Multinomial distribution

outcome: $\{a_0, \cdots a_s\}$

Define $e_0=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}$ $e_1=\begin{pmatrix}0\\1\\\vdots\\0\end{pmatrix}$ $\cdots$ $e_s=\begin{pmatrix}0\\\vdots\\0\\1\end{pmatrix}$ $\in \mathbb{R}^{s+1}$

$Y_i = e_j$ if trial i has outcome a_j. $i=1,\cdots n$. $j=0,\cdots s$.

$Y_1 \cdots Y_n$ — iid. $P(Y_i = e_j) = P_j$. $j=0,\cdots s$.

$$X=\begin{pmatrix}X_0\\X_1\\\vdots\\X_s\end{pmatrix} = \sum_{i=1}^{n} Y_i$$

X_j — number of trials with e_j as the outcome.

$$P(Y_1=y_1, \cdots Y_n=y_n) = \prod_{j=0}^{s} P_j^{x_j} = e^{\sum_{j=0}^{s} x_j \log P_j}.$$

$x = y_1 + \cdots + y_n$.

$s+1$ - parameter exponential family.

not full rank. because $X_1 + \cdots + X_n = n$.

$$P(Y_1=y_1, \cdots Y_n=y_n) = e^{\sum_{i=1}^{s} x_i \log \frac{P_i}{P_0} + n \log P_0}$$

full rank. s-para. exponential family.

$(X_1 \cdots X_s)$ — complete suff.

$$P(X=x) = \sum_{(y_1 \cdots y_n): \sum_{i=1}^{n} y_i = X} P(Y=y) = \sum_{(y_1 \cdots y_n): \sum_{i=1}^{n} y_i = X} \prod_{j=0}^{s} P_j^{x_j}$$

$$= \binom{n}{x_0 \cdots x_s} P_0^{x_0} \cdots P_s^{x_s}$$

$\{1, \cdots n\} \to s+1$ sets

first x_0

next x_1

$\vdots$

$$P(X_0 = x_0, \cdots X_s = x_s) = \binom{n}{x_0, \cdots x_s} P_0^{x_0} \cdots P_s^{x_s}.$$

$X_j \sim Bin(n, P_j)$. $\frac{X_j}{n}$ — UMVUE for P_j

UMVUE for $P_j P_k$.

$$\delta = I(Y_1 = e_j,\ Y_2 = e_k)$$

$$P(\delta = 1 \mid X = x) = \frac{P(\delta=1)\, P(X = x \mid \delta = 1)}{P(X = x)} = \frac{p_j p_k \binom{n-2}{x_0 \cdots x_j - 1, \cdots x_k - 1, \cdots x_s} p_0^{x_0} \cdots p_s^{x_s} \big/ p_j p_k}{\binom{n}{x_0 \cdots x_s} p_0^{x_0} \cdots p_s^{x_s}}$$

$$= \frac{x_j x_k}{n(n-1)}.$$

- Two-way Contingency Tables

Characteristic A: $\{A_1, \cdots A_I\}$

characteristic B: $\{B_1, \cdots B_J\}$.

N_{ij} — number of trials $A_i B_j$.

P_{ij} — chance of $A_i B_j$.

$$N = (N_{11}, \cdots N_{IJ}) \sim \text{Multinomial}(P_{11}, \cdots P_{IJ}; n).$$

$$N_{i+} = \sum_{j=1}^{J} N_{ij} \quad i = 1, \cdots I. \qquad N_{+j} = \sum_{i=1}^{I} N_{ij}$$

Contigency Table:

	B_1 --- B_J	Total
A_1 ⋮ A_I	N_{ij}	N_{i+}
Total	N_{+j}	n

If A and B are independent. $P_{ij} = P_{i+} P_{+j}$

$$P_{i+} = \sum_{j=1}^{J} P_{ij} \quad \text{— chance of } A_i$$

$$P_{+j} = \sum_{i=1}^{I} P_{ij} \quad \text{— chance of } B_j$$

Mass of N:

$$\binom{n}{n_{11}, \cdots n_{IJ}} \prod_{i=1}^{I} \prod_{j=1}^{J} (P_{i+} P_{+j})^{n_{ij}} = \binom{n}{n_{11} \cdots n_{IJ}} \prod_{i=1}^{I} P_{i+}^{n_{i+}} \prod_{j=1}^{J} P_{+j}^{n_{+j}}$$

$$\sum_{i=1}^{I} n_{i+} = \sum_{j=1}^{J} n_{+j} = n$$

$$\sum_{i=1}^{I} P_{i+} = \sum_{j=1}^{J} P_{+j} = 1$$

$$= \binom{n}{n_{11} \cdots n_{IJ}} \exp\left[\sum_{i=2}^{I} n_{i+} \log\left(\frac{P_{i+}}{P_{1+}}\right) + \sum_{j=2}^{J} n_{+j} \log\left(\frac{P_{+j}}{P_{+1}}\right) + n \log(P_{1+} P_{+1}) \right]$$

a full rank $(I+J-2)$-parameter exponential family

Canonical suff. stat.: $(N_{2+}, \cdots N_{I+}, N_{+2}, \cdots N_{+J})$

Equivalent stat. $(N_{1+}, \cdots N_{I+}, N_{+1}, \cdots N_{+J})$.