STAT 417—Midterm

Spring 2016

Name: Solution

Please return these pages with your solutions after exam. Only writing down the final answer will obtain minimum points. Please provide procedure steps.

1. **(10 Points)** Let $X_1, X_2, \ldots, X_n$ be a random sample from a Bernoulli(θ) distribution with parameter $0 \le \theta \le 1$. If $X \sim$ Bernoulli(θ), $\mathbb{P}(X = 1) = \theta$ and $\mathbb{P}(X = 0) = 1 - \theta$. Find the maximum likelihood estimator (MLE) $\hat{\theta}$ of θ. What is the mean squared error (MSE) of $\hat{\theta}$?

3'

Likelihood $L(\theta) = \prod_{i=1}^{n} \theta^{x_i} (1-\theta)^{1-x_i} = \theta^{\sum_{i=1}^{n} x_i} (1-\theta)^{n - \sum_{i=1}^{n} x_i}$

log-likelihood $\ell(\theta) = \sum_{i=1}^{n} x_i \log\theta + (n - \sum_{i=1}^{n} x_i) \log(1-\theta)$

3'

$$\ell'(\theta) = \frac{\sum_i x_i}{\theta} - \frac{n - \sum_i x_i}{1-\theta} = 0$$

$$\hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{n} = \overline{X}.$$

4' — $\mathrm{MSE}(\hat{\theta}) = \underset{2pt}{\text{bias}^2} + \underset{2pt}{Var} = 0 + \frac{\theta(1-\theta)}{n} = \frac{\theta(1-\theta)}{n}$

2. **(10 Points)** Let $X_1, X_2, \ldots, X_n$ be a random sample from a Poisson distribution with parameter λ. The Poisson probability mass function is given by

$$f(x;\lambda) = \frac{\lambda^x e^{-\lambda}}{x!}, \quad x = 0, 1, 2, \ldots, \quad \text{and } \lambda > 0.$$

Both the mean and the variance of a Poisson random variable are λ, i.e., $E(X_1) = \mathrm{Var}(X_1) = \lambda$. If we want to estimate $\theta = \lambda^2$. What is the MLE of θ and check whether it is unbiased. Please justify your answer.

2'

Likelihood $L(\lambda) = \prod_{i=1}^{n} \frac{\lambda^{X_i} e^{-\lambda}}{X_i!} = \frac{\lambda^{\sum X_i} e^{-n\lambda}}{\prod_{i=1}^{n} X_i!}$

log-likelihood $l(\lambda) = \sum_i X_i \log\lambda - n\lambda - \log\prod_{i=1}^{n} X_i!$

3'

MLE: $l'(\lambda) = \frac{\sum X_i}{\lambda} - n = 0$

$\hat{\lambda} = \overline{X}$

2' plug-in $\hat{\theta} = \overline{X}^2$

3' biased: $E\hat{\theta} = E\overline{X}^2 = (E\overline{X})^2 + Var(\overline{X}) = \lambda^2 + \frac{\lambda}{n} \neq \lambda^2$

3. (10 Points) A coin was tossed $n = 1000$ times independently, and we observe 510 heads. Let p be the success probability to obtain a head in one toss.

(a). Do you have evidence to conclude that the coin is fair? Perform a two-sided test to assess $H_0 : p = 0.5$. Report the P-value and your conclusion.

obs. $\frac{X}{n} = \frac{510}{1000} = 0.51$.

3' under H_0: $Z = \dfrac{\frac{X}{n} - 0.5}{\sqrt{\frac{0.5 \cdot 0.5}{1000}}} \sim N(0,1)$ obs. $z = \dfrac{0.51 - 0.5}{\sqrt{\frac{0.5 \cdot 0.5}{1000}}} = 0.6325$

2' p-value $= 2(1 - \Phi(|z|)) = 0.5286$

(b). Construct an approximate 0.95-confidence interval for p.

$$\frac{\frac{X}{n} - p}{\sqrt{\frac{\frac{X}{n}(1-\frac{X}{n})}{n}}} \sim N(0,1)$$

2'

$$0.51 \pm \underset{1.96}{Z_{\frac{1+0.95}{2}}} \sqrt{\frac{0.51 \times 0.49}{1000}} =$$

1'

2'

~~0.03098~~

$(0.47902, 0.54098)$

4. **(15 Points)** A randomly selected sample of $n = 30$ students at a university is asked, "How much did you spend for textbooks this semester?" The responses, in dollars, are

200, 175, 450, 300, 350, 250, 150, 200, 320, 370, 404, $\cdots$, 250.

The sample mean $\bar{x}$ is 284.9 and the sample variance s^2 is 96.1.

5

(a). Construct an **approximate** 90% confidence interval for the population mean using the **central limit theorem.**

$t_{29, 0.95} = 1.6991$

$$\bar{X} \pm Z_{\frac{1+0.9}{2}} \frac{S}{\sqrt{n}} = 284.9 \pm Z_{0.95} \frac{\sqrt{96.1}}{\sqrt{30}}$$

$$= 284.9 \pm 1.645 \sqrt{\frac{96.1}{30}}$$

$$= z \quad (281.9558, 287.844) \qquad 2.9442$$

$$t \quad (281.85897, 287.94103) \qquad 3.04103$$

5

(b). If the data are assumed from $N(\mu, \sigma^2)$, construct an exact 95% confidence interval for the population mean.

$$\bar{X} \pm t_{29, 0.975} \frac{S}{\sqrt{n}}$$

$$= 284.9 \pm 2.0452 \sqrt{\frac{96.1}{30}} \qquad 3.66047$$

$$= (281.2395, 288.5605)$$

5 (c). For Part (b), it is required that the margin error of a 95% confidence interval for μ is no greater than 1 dollar under the same sample variance. What is the required sample size?

$$2.0452 \frac{\sqrt{961}}{\sqrt{n}} \le 1 \qquad n \geq 2.0452^2 \times 961$$

$$= 401.971216$$

5. **(15 Points)** Let $X_1, \ldots X_n$ be a random sample from $N(\mu, \sigma^2)$. The normal pdf is given by

$$f(x; \mu, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < +\infty.$$

(a). If $\mu = 1$ is known, what is the MLE of σ^2?

likelihood: $L(\sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-1)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}} (\sigma^2)^{\frac{n}{2}}} e^{-\frac{\sum (x_i-1)^2}{2\sigma^2}}$

2'

$$l(\sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-1)^2$$

$$l'(\sigma^2) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-1)^2 = 0$$

3'

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-1)^2$$

(b). If μ is unknown, what is the MLE of σ^2?

$2'$

$$L(\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} = \frac{1}{(2\pi)^{\frac{n}{2}}(\sigma^2)^{\frac{n}{2}}} e^{-\frac{\sum_i (x_i-\mu)^2}{2\sigma^2}}$$

$$l(\mu, \sigma^2) = -\frac{n}{2}\log 2\pi - \frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_i (x_i-\mu)^2$$

$3'$

$$\begin{cases} \frac{\partial l}{\partial \mu} = 0 \\ \frac{\partial l}{\partial \sigma^2} = 0 \end{cases} \qquad \begin{aligned} \hat{\mu} &= \overline{X} \\ \hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(X_i - \overline{X})^2 \end{aligned}$$

(c). Are estimators in (a) and (b) unbiased estimators of σ^2? Verify your answer.

$2.5'$ a) $E\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n} E(X_i-1)^2 = \sigma^2$ unbiased!

$2.5'$ b) $E\hat{\sigma}^2 = E\,\frac{n-1}{n}S^2 = \frac{n-1}{n}\sigma^2$ biased!

Bonus. (10 Points) Let $X_1, \ldots, X_n$ be a random sample from the distribution with the pdf $f(x) = 3x^2/\theta^3$, $0 \le x \le \theta$, zero elsewhere. What is the MLE of θ? Is the MLE of θ an unbiased estimator? If no, can you construct an unbiased estimator of θ based on the MLE?

$$L(\theta) = \prod_{i=1}^{n} \frac{3X_i^2}{\theta^3} I_{(0,\theta)}(X_i) = \frac{3^n \prod_{i=1}^{n} X_i^2}{\theta^{3n}} I_{(\max X_i, +\infty)}(\theta) \quad I_{(0,+\infty)}(\min X_i)$$

MLE: $\hat{\theta} = \max(X_1, \cdots X_n)$

$$P(\hat{\theta} \le x) = [P(X_1 \le x)]^n = \left(\int_0^x \frac{3t^2}{\theta^3} dt\right)^n = \frac{x^{3n}}{\theta^{3n}} \quad 0 \le x \le \theta$$

$$f_{\hat{\theta}}(x) = \frac{3n x^{3n-1}}{\theta^{3n}} \quad 0 \le x \le \theta$$

$$E\hat{\theta} = \int_0^\theta x \frac{3n x^{3n-1}}{\theta^{3n}} dx = \frac{3n}{\theta^{3n}} \frac{\theta^{3n+1}}{3n+1} = \frac{3n}{3n+1}\theta \ne \theta$$

$$\frac{3n+1}{3n}\hat{\theta} = \frac{3n+1}{3n}\max(X_1 \cdots X_n) \quad \text{unbiased!}$$

Standard Normal Probabilities

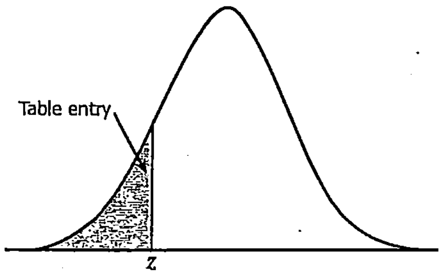
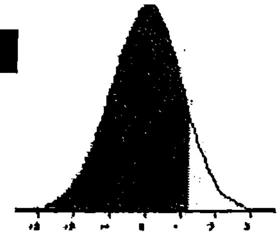


Table entry for z is the area under the standard normal curve to the left of z.

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
−3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
−3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
−3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
−3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
−3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
−2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
−2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
−2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
−2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
−2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
−2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
−2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
−2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
−2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
−2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
−1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
−1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
−1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
−1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
−1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
−1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
−1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
−1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
−1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
−1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
−0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
−0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
−0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
−0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
−0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
−0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
−0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
−0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
−0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
−0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Student's t-distribution table



df	p = 0.75	0.80	0.85	0.90	0.95	0.975	0.980	0.990	0.995	0.9975	0.9990
1	1.0000	1.3764	1.9626	3.0777	6.3137	12.706	15.895	31.821	63.656	127.32	318.29
2	0.8165	1.0607	1.3862	1.8856	2.9200	4.3027	4.8487	6.9645	9.9250	14.089	22.329
3	0.7649	0.9785	1.2498	1.6377	2.3534	3.1824	3.4819	4.5407	5.8408	7.4532	10.214
4	0.7407	0.9410	1.1896	1.5332	2.1318	2.7765	2.9985	3.7469	4.6041	5.5975	7.1729
5	0.7267	0.9195	1.1558	1.4759	2.0150	2.5706	2.7565	3.3649	4.0321	4.7733	5.8935
6	0.7176	0.9057	1.1342	1.4398	1.9432	2.4469	2.6122	3.1427	3.7074	4.3168	5.2075
7	0.7111	0.8960	1.1192	1.4149	1.8946	2.3646	2.5168	2.9979	3.4995	4.0294	4.7853
8	0.7064	0.8889	1.1081	1.3968	1.8595	2.3060	2.4490	2.8965	3.3554	3.8325	4.5008
9	0.7027	0.8834	1.0997	1.3830	1.8331	2.2622	2.3984	2.8214	3.2498	3.6896	4.2969
10	0.6998	0.8791	1.0931	1.3722	1.8125	2.2281	2.3593	2.7638	3.1693	3.5814	4.1437
11	0.6974	0.8755	1.0877	1.3634	1.7959	2.2010	2.3281	2.7181	3.1058	3.4966	4.0248
12	0.6955	0.8726	1.0832	1.3562	1.7823	2.1788	2.3027	2.6810	3.0545	3.4284	3.9296
13	0.6938	0.8702	1.0795	1.3502	1.7709	2.1604	2.2816	2.6503	3.0123	3.3725	3.8520
14	0.6924	0.8681	1.0763	1.3450	1.7613	2.1448	2.2638	2.6245	2.9768	3.3257	3.7874
15	0.6912	0.8662	1.0735	1.3406	1.7531	2.1315	2.2485	2.6025	2.9467	3.2860	3.7329
16	0.6901	0.8647	1.0711	1.3368	1.7459	2.1199	2.2354	2.5835	2.9208	3.2520	3.6861
17	0.6892	0.8633	1.0690	1.3334	1.7396	2.1098	2.2238	2.5669	2.8982	3.2224	3.6458
18	0.6884	0.8620	1.0672	1.3304	1.7341	2.1009	2.2137	2.5524	2.8784	3.1966	3.6105
19	0.6876	0.8610	1.0655	1.3277	1.7291	2.0930	2.2047	2.5395	2.8609	3.1737	3.5793
20	0.6870	0.8600	1.0640	1.3253	1.7247	2.0860	2.1967	2.5280	2.8453	3.1534	3.5518
21	0.6864	0.8591	1.0627	1.3232	1.7207	2.0796	2.1894	2.5176	2.8314	3.1352	3.5271
22	0.6858	0.8583	1.0614	1.3212	1.7171	2.0739	2.1829	2.5083	2.8188	3.1188	3.5050
23	0.6853	0.8575	1.0603	1.3195	1.7139	2.0687	2.1770	2.4999	2.8073	3.1040	3.4850
24	0.6848	0.8569	1.0593	1.3178	1.7109	2.0639	2.1715	2.4922	2.7970	3.0905	3.4668
25	0.6844	0.8562	1.0584	1.3163	1.7081	2.0595	2.1666	2.4851	2.7874	3.0782	3.4502
26	0.6840	0.8557	1.0575	1.3150	1.7056	2.0555	2.1620	2.4786	2.7787	3.0669	3.4350
27	0.6837	0.8551	1.0567	1.3137	1.7033	2.0518	2.1578	2.4727	2.7707	3.0565	3.4210
28	0.6834	0.8546	1.0560	1.3125	1.7011	2.0484	2.1539	2.4671	2.7633	3.0470	3.4082
29	0.6830	0.8542	1.0553	1.3114	1.6991	2.0452	2.1503	2.4620	2.7564	3.0380	3.3963
30	0.6828	0.8538	1.0547	1.3104	1.6973	2.0423	2.1470	2.4573	2.7500	3.0298	3.3852
31	0.6825	0.8534	1.0541	1.3095	1.6955	2.0395	2.1438	2.4528	2.7440	3.0221	3.3749
32	0.6822	0.8530	1.0535	1.3086	1.6939	2.0369	2.1409	2.4487	2.7385	3.0149	3.3653
33	0.6820	0.8526	1.0530	1.3077	1.6924	2.0345	2.1382	2.4448	2.7333	3.0082	3.3563
34	0.6818	0.8523	1.0525	1.3070	1.6909	2.0322	2.1356	2.4411	2.7284	3.0020	3.3480
35	0.6816	0.8520	1.0520	1.3062	1.6896	2.0301	2.1332	2.4377	2.7238	2.9961	3.3400
36	0.6814	0.8517	1.0516	1.3055	1.6883	2.0281	2.1309	2.4345	2.7195	2.9905	3.3326
37	0.6812	0.8514	1.0512	1.3049	1.6871	2.0262	2.1287	2.4314	2.7154	2.9853	3.3256
38	0.6810	0.8512	1.0508	1.3042	1.6860	2.0244	2.1267	2.4286	2.7116	2.9803	3.3190
39	0.6808	0.8509	1.0504	1.3036	1.6849	2.0227	2.1247	2.4258	2.7079	2.9756	3.3127
40	0.6807	0.8507	1.0500	1.3031	1.6839	2.0211	2.1229	2.4233	2.7045	2.9712	3.3069
50	0.6794	0.8489	1.0473	1.2987	1.6759	2.0086	2.1087	2.4033	2.6778	2.9370	3.2614
60	0.6786	0.8477	1.0455	1.2958	1.6706	2.0003	2.0994	2.3901	2.6603	2.9146	3.2317
75	0.6778	0.8464	1.0436	1.2929	1.6654	1.9921	2.0901	2.3771	2.6430	2.8924	3.2024
100	0.6770	0.8452	1.0418	1.2901	1.6602	1.9840	2.0809	2.3642	2.6259	2.8707	3.1738
∞	0.6745	0.8416	1.0364	1.2816	1.6449	1.9600	2.0537	2.3263	2.5758	2.8070	3.0902