STAT 417 Lecture NOTE 11

Let $\hat{\theta}_{ML}$ be the MLE of θ. So the plug-in MLE for estimating $\psi = r(\theta)$ is $\hat{\psi} = r(\hat{\theta}_{ML})$. Is $\hat{\psi}$ good or not?

there are two ways to define goodness:

First way: accuracy

Second way: reliability.

The <u>biasness</u> of $\hat{\psi}$. is defined as $E(\hat{\psi}) - \psi$.

$\hat{\psi}$ is said to be <u>unbiased</u> if $E(\hat{\psi}) - \psi = 0$.

Eg1. If $X_1, X_2, \cdots X_n$ are iid samples from $N(\mu, \sigma^2)$.

the MLE of μ is $\hat{\mu}_{ML} = \bar{X}$. then. $\hat{\mu}_{ML}$ is unbiased.

proof: $E(\hat{\mu}_{ML}) = E(\bar{X}) = E(\frac{1}{n}(X_1 + \cdots + X_n)) = \frac{1}{n}[E(X_1) + \cdots + E(X_n)] = \mu$.

so $\hat{\mu}_{ML}$ is unbiased.

<u>E.g2</u>. If X follows ~~Bernoulli~~ Binomial(n, p). the MLE of p is $\hat{p}_{ML} = \frac{X}{n}$. Since $E(\hat{p}_{ML}) = E(\frac{X}{n}) = \frac{1}{n}E(X) = \frac{1}{n} \cdot np = p$.

$\hat{p}_{ML}$ is unbiased. To estimate ~~the plug-in MLE~~ p^2. the plug-in MLE is $\hat{p}^2_{ML} = (\frac{X}{n})^2$. Next we show that $E(\hat{p}^2_{ML}) \neq p^2$. so $\hat{p}^2_{ML}$ is NOT an unbiased estimate of p^2.

Since $E(X) = np$.

$$E(X^2) = Var(X) + (E(X))^2 = np(1-p) + (np)^2.$$

We get

$$E(\hat{p}_{ML}^2) = E\left(\left(\frac{X}{n}\right)^2\right) = \frac{1}{n^2} E(X^2) = \frac{1}{n^2}\left(n^2p^2 + np(1-p)\right)$$

$$= p^2 + \frac{1}{n} p(1-p) \neq p^2.$$

The mean squared error (MSE) of $\hat{\psi}$ is

$$MSE_\psi(\hat{\psi}) = E\left((\hat{\psi} - \psi)^2\right).$$

Eg 3. If $X_1, X_2,$ are iid $\sim N(\mu, \sigma^2)$. The MLE of μ is $\hat{\mu}_{ML} = \bar{X}$. Find $MSE_\mu(\hat{\mu}_{ML})$.

Sol. $MSE_\mu(\hat{\mu}_{ML}) = E\left((\bar{X} - \mu)^2\right)$

$$= E\left(\left(\frac{X_1+X_2}{2} - \mu\right)^2\right) = E\left(\left[\frac{1}{2}(X_1-\mu) + (X_2-\mu)\right]^2\right)$$

$$= \frac{1}{4} E\left(\left[(X_1-\mu) + (X_2-\mu)\right]^2\right)$$

$$= \frac{1}{4}\left\{E\left((X_1-\mu)^2\right) + 2E\left((X_1-\mu)(X_2-\mu)\right) + E\left((X_2-\mu)^2\right)\right\}$$

$$= \frac{1}{4}(\sigma^2 + 0 + \sigma^2) = \frac{1}{2}\sigma^2.$$

Eg 4. Let $X_1, X_2, \dots, X_n$ be iid $\sim N(\mu, \sigma^2)$. Find $MSE_\mu(\hat{\mu}_{ML})$.

sol: $\hat{\mu}_{ML} = \bar{X}$, so

$$MSE_\mu(\hat{\mu}_{ML}) = E\left((\bar{X} - \mu)^2 \right) = E\left(\left(\frac{X_1 + \dots + X_n}{n} - \mu \right)^2 \right)$$

$$= E\left(\left| \frac{1}{n}\left[(X_1 - \mu) + (X_2 - \mu) + \dots + (X_n - \mu) \right] \right|^2 \right)$$

$$= \frac{1}{n^2} E\left(\left[(X_1 - \mu) + (X_2 - \mu) + \dots + (X_n - \mu) \right]^2 \right)$$

$$= \frac{1}{n^2} E\Big((X_1 - \mu)^2 + (X_2 - \mu)^2 + \dots + (X_n - \mu)^2$$

$$+ 2(X_1 - \mu)(X_2 - \mu) + 2(X_1 - \mu)(X_3 - \mu) + \dots + 2(X_{n-1} - \mu)(X_n - \mu) \Big)$$

$$= \frac{1}{n^2}\left[E\left((X_1 - \mu)^2 \right) + \dots + E\left((X_n - \mu)^2 \right) \right]$$

$$= \frac{1}{n^2} \cdot n\sigma^2 = \frac{1}{n}\sigma^2.$$

Interestingly, $n \to \infty$ yields $MSE_\mu(\hat{\mu}_{ML}) = \frac{1}{n}\sigma^2 \to 0$.

EX2 ~~Ex3~~. If X is a sample from Binomial (n, p).

The MLE of p is $\hat{p}_{ML} = \frac{X}{n}$. Find $MSE_p(\hat{p}_{ML})$. what happens if $n \to \infty$?

$$MSE_p(\hat{p}_{ML}) = E\left(\left(\frac{X}{n} - p\right)^2 \right)$$

$$= E\left(\left(\frac{X}{n}\right)^2\right) - 2p\,E\left(\frac{X}{n}\right) + p^2$$

$$= \frac{1}{n^2}\left(n^2p^2 + np(1-p)\right) - 2p^2 + p^2$$

$$= \frac{1}{n}p(1-p). \to 0 \text{ as } n \to \infty.$$

EX0. If $X_1, \dots, X_n$ ~~X~~ $\sim$ Bernoulli ~~Exponential~~ (θ). The MLE of θ is $\hat{\theta}_{ML} = \bar{X}$.

Is $\hat{\theta}_{ML}$ unbiased? Note: $X \sim$ Bernoulli (θ) if $P(X=1) = 1 - P(X=0) = \theta$.

If we want to estimate θ^2. Is ~~your~~ $\hat{\theta}_{ML}^2$ unbiased?

Sol: $E(\hat{\theta}_{ML}) = E(\bar{X}) = \frac{1}{n}\left(E(X_1) + \dots + E(X_n)\right) = \frac{1}{n} \cdot n\theta = \theta$

Note $X_1 + \dots + X_n = \#$ of 1s. $\sim$ Binomial (n, θ).

so. $\bar{X}$ is not unbiased.

EX1. If X is a sample from poisson (θ). ① Is $\hat{\theta}_{ML}$ unbiased?

$E(\hat{\theta}_{ML}) = E(X) = \theta$. so $\hat{\theta}$ is unbiased.

② If we want to estimate θ^2. Is $\hat{\theta}_{ML}^2$ unbiased?

$E(\hat{\theta}_{ML}^2) = E(X^2) = Var(X) + (E(X))^2 = \theta + \theta^2 \neq \theta^2$. so unbiased.