posterior distribution under independent samples.

Eg 7. If $X_1, X_2, \ldots, X_n$ are i.i.d samples following the Bernoulli models:

$$p(X_i = 1) = 1 - p(X_i = 0) = \theta.$$

Assume that θ has the following prior distribution:

$$f(\theta) = \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1}, \quad 0 \le \theta \le 1. \quad (\text{Beta distribution})$$

then the posterior distribution of θ is

$$f(\theta \mid x_1, \ldots, x_n) \propto f_\theta(x_1, \ldots, x_n)\, f(\theta)$$

$$= f_\theta(x_1) \cdots f_\theta(x_n)\, f(\theta)$$

$$= \theta^{x_1}(1-\theta)^{1-x_1}\, \theta^{x_2}(1-\theta)^{1-x_2} \cdots \theta^{x_n}(1-\theta)^{1-x_n}\, f(\theta)$$

$$= \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})} \cdot \frac{1}{B(\alpha, \beta)} \theta^{\alpha-1}(1-\theta)^{\beta-1}.$$

$$= \frac{1}{B(\alpha, \beta)} \theta^{n\bar{x}+\alpha-1} (1-\theta)^{n(1-\bar{x})+\beta-1}.$$

so the posterior distribution of θ is $\text{Beta}(n\bar{x}+\alpha,\ n(1-\bar{x})+\beta)$.

Eg 8. Let $X_1, \cdots, X_n$ be i.i.d samples following $N(\mu, \sigma_0^2)$.

where σ_0^2 is known. $\mu \in \mathbb{R}$ is unknown. The prior for μ is

$\mu \sim N(\mu_0, \tau_0^2)$, where both μ_0 and τ_0^2 are known.

Then ~~the~~

$$f(\mu \mid x_1, x_2, \cdots, x_n) \propto f(x_1, \cdots, x_n \mid \mu)\, f(\mu)$$

$$= \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n e^{-\frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma_0^2}} \cdot \frac{1}{\sqrt{2\pi}\,\tau_0} e^{-\frac{(\mu-\mu_0)^2}{2\tau_0^2}}$$

$$= \exp\left(-\frac{(n\mu^2 - 2n\bar{x}\mu)\tau_0^2 + \mu^2\sigma_0^2 - 2\mu_0\sigma_0^2\mu}{2\sigma_0^2\tau_0^2}\right) \cdot \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n \cdot \frac{1}{\sqrt{2\pi}\,\tau_0} \exp\left(-\frac{\sum x_i^2}{2\sigma_0^2}\right) \cdot \exp\left(-\frac{\mu_0^2}{2\tau_0^2}\right)$$

$$= \exp\left(-\frac{(n\tau_0^2+\sigma_0^2)\mu^2 - 2(n\bar{x}\tau_0^2 + \mu_0\sigma_0^2)\mu}{2\sigma_0^2\tau_0^2}\right) \cdot \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n \frac{1}{\sqrt{2\pi}\,\tau_0} \exp\left(-\frac{\sum x_i^2}{2\sigma_0^2}\right) \cdot \exp\left(-\frac{\mu_0^2}{2\tau_0^2}\right)$$

$$= \exp\left(-\frac{(n\tau_0^2+\sigma_0^2)\left(\mu - \frac{n\bar{x}\tau_0^2 + \mu_0\sigma_0^2}{n\tau_0^2+\sigma_0^2}\right)^2}{2\sigma_0^2\tau_0^2}\right)$$

$$\cdot \left(\frac{1}{\sqrt{2\pi}\,\sigma_0}\right)^n \frac{1}{\sqrt{2\pi}\,\tau_0} \exp\left(-\frac{\sum x_i^2}{2\sigma_0^2}\right) \cdot \exp\left(-\frac{\mu_0^2}{2\tau_0^2}\right) \cdot \exp\left(-\frac{(n\bar{x}\tau_0^2 + \mu_0\sigma_0^2)}{2\sigma_0^2\tau_0^2(n\tau_0^2+\sigma_0^2)}\right)$$

So μ has posterior distribution $N\left(\frac{n\bar{x}\tau_0^2 + \mu_0\sigma_0^2}{n\tau_0^2+\sigma_0^2}, \frac{\sigma_0^2\tau_0^2}{n\tau_0^2+\sigma_0^2}\right)$

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an exercise: If $\bar{x} = 0$, $n = 100$, $\mu_0 = 0$, $\sigma_0^2 = 200$, $\tau_0^2 = 2$.

then the posterior distribution of μ is $N(0, 1)$. find mean and variance of μ given s.

Answer: 0 and 1. find $P(\mu < 1 \mid s) = 0.8413$.

~~Eg 9~~ EX1. Suppose $X_1, X_2 \cdots X_n$ follow Exponential (θ).

while, θ has prior Exponential (θ_0). $\theta_0 > 0$ is known.

Find the posterior distribution of θ.

Sol: $f(\theta | x_1, \cdots, x_n) \propto f_\theta(x_1, \cdots x_n) f(\theta)$

$$\propto \theta e^{-\theta x_1} \cdots \theta e^{-\theta x_n} \cdot \theta_0 e^{-\theta_0 \theta}.$$

$$\propto \theta_0 \cdot \theta^n e^{-(n\bar{x} + \theta_0)\theta}.$$

so the posterior of θ is ~~$\theta \sim$ Exponential(~~ $f(\theta | x_1 \cdots x_n)$ as above.

clearly, $\theta | x_1 \cdots x_n \sim \text{Gamma}(n+1, \; n\bar{x} + \theta_0)$.

posterior mean and variance:

suppose θ given the sample s has posterior distribution $f(\theta|s)$.

Then the posterior mean of θ is

$$E(\theta|s) = \int \theta f(\theta|s)\, d\theta.$$

the posterior variance of θ is

$$\mathrm{Var}(\theta|s) = \int (\theta - E(\theta|s))^2 f(\theta|s)\, d\theta.$$

$$= \int \theta^2 f(\theta|s)\, d\theta - [E(\theta|s)]^2.$$

Eg 9. In Eg 7. the posterior distribution of θ given $X_1, \ldots, X_n$ is

$$f(\theta|X_1, \ldots, X_n) \propto \theta^{n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1} \quad \cancel{\theta^{\alpha-1}(1-\theta)^{\beta-1}}, \quad 0 \le \theta \le 1.$$

clearly. $\theta \sim \mathrm{Beta}(n\bar{x}+\alpha,\ n(1-\bar{x})+\beta)$.

Find the posterior mean and variance of θ.

sol: $\mathrm{Beta}(a, b)$ has mean $\dfrac{a}{a+b}$,

variance $\dfrac{ab}{(a+b+1)(a+b)^2}$.

so. posterior mean $= \dfrac{n\bar{x}+\alpha}{n+\alpha+\beta}$.

posterior variance $= \dfrac{(n\bar{x}+\alpha)(n(1-\bar{x})+\beta)}{(n+\alpha+\beta+1)(n+\alpha+\beta)^2}$

Eg10. In Eg 8. Let $X_1, \dots, X_n$ be i.i.d samples from $N(\mu, \sigma_0^2)$, the prior distribution on μ is $N(\mu_0, \tau_0^2)$. where $\sigma_0^2, \mu_0, \tau_0^2$ are known. Then we obtained the posterior distribution of μ : $N\left(\dfrac{n\bar{X}\tau_0^2 + \mu_0\sigma_0^2}{n\tau_0^2 + \sigma_0^2}, \dfrac{\sigma_0^2 \tau_0^2}{n\tau_0^2 + \sigma_0^2} \right)$

So the posterior mean of μ is $\dfrac{n\bar{x}\,\tau_0^2 + \mu_0\tau_0^2}{n\tau_0^2 + \sigma_0^2}$.

the posterior variance of μ is $\dfrac{\sigma_0^2\tau_0^2}{n\tau_0^2 + \sigma_0^2}$.

EX 2. In Ex1. $X_1, \dots, X_n$ are i.i.d samples from Exponential (θ). while θ follows prior Exponential $\theta_0)$, where $\theta_0 > 0$ is known.

Find the posterior mean and variance of θ.

Clearly, $\theta \mid X_1, \dots X_n \sim \Gamma\text{amma}(n+1, n\bar{x} + \theta_0)$.

So the posterior mean of θ is $\dfrac{n+1}{n\bar{x} + \theta_0}$,

the posterior variance $= \dfrac{n+1}{(n\bar{x} + \theta_0)^2}$

Eg 11. Let $X_1, \ldots, X_n$ be i.i.d samples from $\mathrm{Unif}(0, \theta)$.

the prior distribution of θ be ~~[illegible]~~. Exponential(α).

- Find the posterior distribution of θ.

Sol: $$f_\theta(x_1 \ldots x_n) = \frac{1}{\theta}\mathbb{1}(0 < x_1 < \theta) \cdots \frac{1}{\theta}\mathbb{1}(0 < x_n < \theta)$$

$$= \frac{1}{\theta^n}\mathbb{1}(0 < x_1 < \theta) \cdots \mathbb{1}(0 < x_n < \theta)$$

$$= \begin{cases} \theta^{-n}, & \text{if } \cancel{\theta > x_1, \ldots, x_n}. \ \theta > \max\{x_1, \ldots, x_n\}. \\ 0, & \text{o/w}. \end{cases}$$

$$= \theta^{-n} I_{\theta > \max(x_1, \ldots, x_n)}.$$

So. the posterior distribution of θ is

$$f_{\theta|}(x_1 \ldots x_n) \propto f_\theta(x_1 \ldots x_n) f(\theta) = \theta^{-n}\alpha e^{-\alpha\theta} I_{\theta > \max(x_1, \ldots, x_n)}.$$

$$= \alpha\,\theta^{-n} e^{-\alpha\theta} I_{\theta > \max(x_1, \ldots, x_n)}.$$