Lecture Note 12.

Inference on moments:

Let X be a random variable whose jth moment is

$$\mu_j = E(X^j), \quad j = 1, 2, 3, \cdots$$

Question: ① how to estimate μ_j?

② how to find CI for μ_j?

③ how to test hypothesis on μ_j?

Suppose $X_1, X_2, \cdots, X_n$ are i.i.d samples of X, but we don't know their distributions. Let us answer the above questions ①–③.

A natural estimation of μ_j is the sample moment:

$$m_j = \frac{1}{n}\sum_{i=1}^{n} X_i^j, \quad j=1, 2, 3, \cdots$$

Now $X_1^j, X_2^j, \cdots, X_n^j$ are i.i.d samples with $\mu_j = E(X^j)$.

$$\sigma_j^2 = Var(X^j) = E((X^j)^2) - [E(X^j)]^2.$$

$$= E(X^{2j}) - \mu_j^2 = \mu_{2j} - \mu_j^2.$$

So By CLT,

$$\sqrt{n}\left(\frac{m_j - \mu_j}{\sigma_j}\right) \overset{\sim}{\approx} Z \sim N(0,1).$$

The 95%-CI for μ_j is thus obtained by the following.

first set

$$-1.96 \leq \sqrt{n} \left(\frac{m_j - \mu_j}{\sigma_j} \right) \leq 1.96$$

then.

$$m_j - 1.96 \frac{\sigma_j}{\sqrt{n}} \leq \mu_j \leq m_j + 1.96 \frac{\sigma_j}{\sqrt{n}}$$

so. the 95%-CI is

$$m_j \pm 1.96 \frac{\sigma_j}{\sqrt{n}}.$$

σ_j is unknown since it depends on unknowns μ_{2j} and μ_j. so we cheat on them by replacing them by m_{2j} and m_j. so.

the 95%-CI for μ_j is

$$m_j \pm 1.96 \frac{s_j}{\sqrt{n}},$$

where $s_j = \sqrt{s_j^2}$ and $s_j^2 = m_{2j} - m_j^2$.

In general, the γ-CI for μ_j is

$$m_j \pm z_{\frac{1+\gamma}{2}} \cdot \frac{s_j}{\sqrt{n}}.$$

Eg 1. Suppose the salaries of stat employee are (in $\$10000$)

1, 2, 1.5, 1.3, 1.6, 0.9, 3.3, 4.2, 2.7, 1.1

what is an estimate of μ_3 ?

sol: $m_3 = \frac{1}{10}\left(1^3 + 2^3 + 1.5^3 + 1.3^3 + 1.6^3 + 0.9^3 + 3.3^3 + 4.2^3 + 2.7^3 + 1.1^3\right)$

$= 15.0436.$

Eg 2. In eg 1. establish the 95% CI of μ_3.

sol: $m_6 = \frac{1}{10}\left(1^6 + 2^6 + 1.5^6 + 1.3^6 + 1.6^6 + 0.9^6 + 3.3^6 + 4.2^6 + 2.7^6 + 1.1^6\right)$

$= 726.8218.$

so $s_3^2 = m_6 - m_3^2 = 726.8218 - 15.0436^2 = 500.5119$

so $s_3 = \sqrt{s_3^2} = 22.37212.$

so the 95%-CI for μ_3 is

$$15.0436 \pm 1.96 \frac{22.37212}{\sqrt{10}} = [1.17722,\ 28.90998].$$

Test $H_0: \mu_j = \mu_j^0$. we use the following z-test.

$$z = \sqrt{n}\left(\frac{m_j - \mu_j^0}{s_j}\right).$$

The p-value is $2(1-\Phi(|z|))$.

<u>E.g.3.</u> In E.g 1. Test $H_0: \mu_3 = 0$.

$$z = \sqrt{10}\left(\frac{15.0436 - 0}{22.37212}\right) = 2.12.$$

so. the p-value is $2(1-\Phi(2.12)) = 0.034 < 0.05$.

so $H_0: \mu_3 = 0$ is rejected at 0.05.