STAT 417. lecture 2.

Discrete distribution:

① Binomial distribution: n experiments with only two outcomes. F and S.

X = number of successes in n experiments.

$P(S) = p$.

Then $X \sim \text{Binomial}(n, p)$. The distribution function is

$$f(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}.$$

Fact: $E(X) = np$. $\text{Var}(X) = np(1-p)$.

E.g. 6. ~~P(exactly two sixs out of five rolls of a fair~~

Roll a fair die five times. X = # of sixes.

consider success = six.

failure = none six.

then $p = P(\text{success}) = \frac{1}{6}$. so $X \sim \text{Binomial}(5, \frac{1}{6})$.

$$P(X = 2) = \binom{5}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^3 = 0.161.$$

②. poisson distribution. X is poisson if

$$f(x) = P(X = x) = \frac{\lambda^x}{x!} e^{-\lambda}, \quad \lambda > 0, \quad x = 0, 1, 2, \cdots.$$

Fact: $E(X) = \text{Var}(X) = \lambda$.

E.g. 7 ~~we can ideally consider the following random variables as poisson~~

~~London has 576 regions. 500 of them with attacks.~~

~~so each region has $p = 1 - \frac{500}{576} = 0.132$ to be not attacked.~~

cdf and pdf. if X is continuous, then the cdf of X is

$$F(x) = P(X \le x).$$

the pdf of X is $f(x) = \frac{d}{dx} F(x)$.

properties of cdf: ① $F(-\infty) = 0$. $F(\infty) = 1$.

② F is nondecreasing.

③. $0 \le F(x) \le 1$. for all $x \in \mathbb{R}$.

Expectation and variance:

Discrete:

$$E(X) = \sum_{\text{all } x} x f(x) \qquad Var(X) = \sum_{\text{all } x} (x - E(X))^2 f(x).$$

Continuous: $E(X) = \int x f(x)\, dx$. $Var(X) = \int (x - E(X))^2 f(x)\, dx$.

property: $Var(X) = E(X^2) - (E(X))^2$.

London has 576 regions, each one has probability $p = 0.002$ to be attacked. Let X = # of attacked regions in London.

Then $X \sim \text{Binomial}(576, 0.002) \approx \text{poisson}(1.152)$.

$\Uparrow$ poisson approximation to Binomial.

Continuous distribution:

① uniform distribution: $X \sim \text{unif}(a, b)$ if

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0. & \text{o/w.} \end{cases}$$

$$F(x) = P(X \le x) = \begin{cases} 0. & x \le a. \\ \frac{x-a}{b-a}, & a \le x \le b. \\ 1, & x > b. \end{cases}$$

$$E(X) = \frac{a+b}{2}, \quad \text{Var}(X) = \frac{(b-a)^2}{12}.$$

②. Exponential distribution: $X \sim \text{Exponential}(\alpha)$ if

$$f(x) = \begin{cases} \alpha e^{-\alpha x}. & x \geq 0 \\ 0. & \text{o/w.} \end{cases}$$

$$F(x) = \begin{cases} 1 - e^{-\alpha x}. & x \geq 0. \\ 0. & x < 0. \end{cases}$$

$$E(X) = \alpha^{-1}, \quad \text{Var}(X) = \alpha^{-2}.$$

③. Gamma distribution: $X \sim \text{Gamma}(\alpha, \beta)$. if

$$f(x) = \begin{cases} \dfrac{x^{\alpha-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha)\beta^{\alpha}}, & x \geq 0. \\ 0, & x < 0 \end{cases}$$

$E(X) = \alpha\beta$, $\quad Var(X) = \alpha\beta^2$.

④. Normal distribution: $X \sim N(\mu, \sigma^2)$ if

$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, \qquad -\infty < x < \infty.$$

$E(X) = \mu$, $\quad Var(X) = \sigma^2$.

Define the cdf of $\underbrace{\cancel{x}}_{X \sim N(0,1)}$ to be $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\, dt$.

$\Phi(x)$ has no closed form.

Function of random variable.

If X has pdf $f_X(x)$, ~~then~~ $r(x)$ is a monotone function of X, whose inverse function is $s(y)$.

then $Y = r(X)$ has pdf.

$$f_Y(y) = f_X(s(y))\,|s'(y)|.$$

E.g. 8: $Y = \log X$, then $s(y) = e^y$. so $s'(y) = e^y$.

$$f_Y(y) = f_X(e^y)\,e^y.$$

Joint ~~Conditional~~ distribution: $F(x,y) = P(X \le x \text{ and } Y \le y)$. (cdf).

$$\cancel{f(x,y) = \frac{d^2}{dx\,dy} F(x,y)}$$

discrete ~~distribution~~ situation:

distribution function: $f(x,y) = P(X = x \text{ and } Y = y)$.

conditional distribution function:

$$f(x|y) = \frac{f(x,y)}{f(y)} = \frac{f(x,y)}{\sum_{\text{all } x} f(x,y)}.$$

Continuous situation:

~~distribution function~~ joint pdf: $f(x,y) = \frac{d^2}{dx\,dy} F(x,y)$.

conditional pdf: $f(x|y) = \frac{f(x,y)}{f(y)} = \frac{f(x,y)}{\int f(x,y)\,dx}$

Conditional expectation: $E(X|Y=y) = \int x\, f(x|y)\,dx \equiv \mu_{x|y}$.

conditional variance: $var(X|Y=y) = \cancel{\int x^2 f(x|y)\,dx - \left(\int x f(x|y)\,dx\right)^2}$

$$\int (x - \mu_{x|y})^2 f(x|y)\,dx$$

$$= \int x^2 f(x|y)\,dx - \mu_{x|y}^2$$