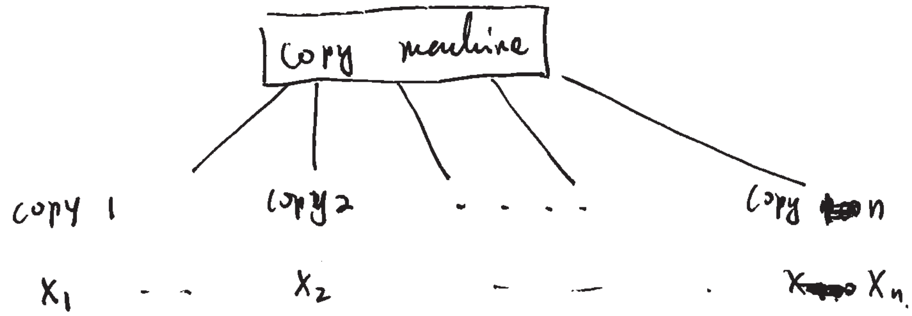
STAT 417 lecture Note 4.

① i.i.d = independent and identically distributed.



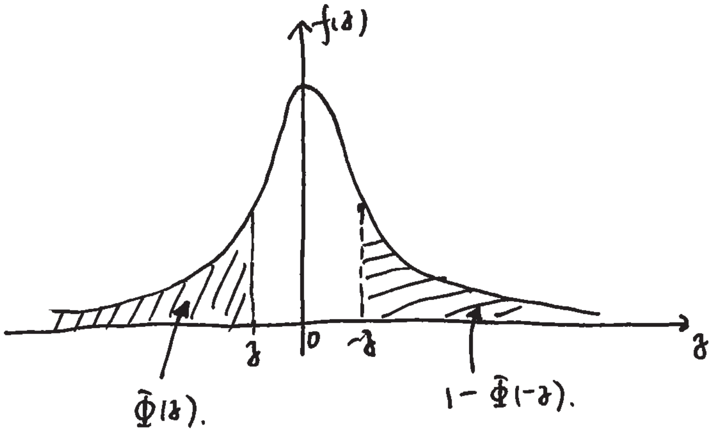
$X_1, X_2, \ldots, X_n$ are independent and identically distributed.

or simply, $X_1, \ldots, X_n$ are i.i.d.

② standard normal distribution: $Z \sim N(0,1)$ if Z has pdf

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \quad -\infty < z < \infty.$$

cdf of Z is $\Phi(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{t^2}{2}} dt$.



By symmetry, $\Phi(z) = 1 - \Phi(-z)$. for any z.

theorem 4. For any $z \in \mathbb{R}$. $\Phi(-z) = 1 - \Phi(z)$.

E.g. 1. let $X_1, X_2, \cdots, X_{36}$ be iid poisson (4).

let $\bar{X} = \dfrac{X_1 + X_2 + \cdots + X_{36}}{36}$, then what is $P(\bar{X} < 4.2)$?

sol: Since by central limit theorem,

$$\sqrt{36}\left(\frac{\bar{X}-4}{2}\right) \approx Z.$$

$$P(\bar{X} < 4.2) = P\left(\sqrt{36}\left(\frac{\bar{X}-4}{2}\right) < \sqrt{36}\left(\frac{4.2-4}{2}\right)\right) \approx P(Z < 0.6)$$

$$= 1 - \Phi(-0.6) = 1 - 0.2743 = 0.7257$$

Central Limit theorem: (Thm 5.) Suppose $X_1, X_2, \ldots, X_n$ are iid with $E(X_i)=\mu$, $\text{var}(X_i)=\sigma^2<\infty$. ~~Then~~. let $\bar{X} = \dfrac{X_1+X_2+\cdots+X_n}{n}$ be the sample mean.

Then. $$\sqrt{n}\left(\frac{\bar{X}-\mu}{\sigma}\right) \approx Z, \text{ where } Z \sim N(0,1).$$

E.g. 2. suppose $X_1, X_2, \ldots, X_{100}$ are iid with the following distribution.

$$P(X_i=1) = 1-P(X_i=0) = 0.6.$$

So. $E(X_i)=\mu=0.6$, $\text{Var}(X_i)=\sigma^2=(0.6)(0.4)=0.24$

So. $P(0.55<\bar{X}<0.65)$

$$= P\left(\sqrt{100}\left(\frac{0.55-0.6}{\sqrt{0.24}}\right) < \sqrt{100}\left(\frac{\bar{X}-0.6}{\sqrt{0.24}}\right) < \sqrt{100}\left(\frac{0.65-0.6}{\sqrt{0.24}}\right)\right)$$

$$\approx P(-1.02<Z<1.02) = \Phi(1.02)-\Phi(-1.02)$$

See page 712. Table D.2.

$\Phi(-1.02)=0.1539$. $\Phi(1.02)=1-\Phi(-1.02)=0.8461$.

So. $P(0.55<\bar{X}<0.65) \approx 0.8461-0.1539=0.6922$.

E.X. 2. Let $X_1, X_2, \ldots, X_{400}$ be iid with Exponential(3).

Use central limit theorem to find ① $P(X_1+X_2+\cdots+X_{400} < \cancel{136}\ 135)$

Sol: $\cancel{P(X_1+\cdots+X_{400}<130) = P(\bar{X} < 0.325)}$ $\cancel{0.3375}$ $\cancel{0.34}$

$$\cancel{= P\left(20\left(\frac{\bar{X}-\frac{1}{3}}{\frac{1}{3}}\right) < 20\left(\frac{0.325-\frac{1}{3}}{\frac{1}{3}}\right)\right) \approx P(Z<0.25)}$$ 0.3375 $\cancel{0.34}$

$$\cancel{= \Phi(0.25) = 1-\Phi(-0.25) = 1-0.4013 = 0.5987.}$$

(ii) $P(133 < X_1 + X_2 + \cdots + X_{400} < 135) = P(0.3325 < \bar{X} < 0.3375)$

$$= P\left(20\left(\frac{0.3325 - \frac{1}{3}}{\frac{1}{3}} \right) < 20\left(\frac{\bar{X} - \frac{1}{3}}{\frac{1}{3}} \right) < 20\left(\frac{0.3375 - \frac{1}{3}}{\frac{1}{3}} \right) \right)$$

$$\approx P(-0.05 < Z < 0.25)$$

$$= \Phi(0.25) - \Phi(-0.05) = 1 - \Phi(-0.25) - \Phi(-0.05)$$

$$= 1 - 0.4013 - 0.4801 = 0.1186.$$