STAT 417 lecture NOTE 15.

Eg 12: (multinomial model). let the sample $(X_1, \dots, X_k)$ be from multinomial model with parameter $\theta = (\theta_1, \dots, \theta_k)$, where θ is unknown, $\theta_1, \dots, \theta_k \geq 0$. $\theta_1 + \cdots + \theta_k = 1$.

let θ have prior distribution

$$f(\theta_1, \theta_2, \dots, \theta_k) = \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1}, \quad \alpha_1 \cdots \alpha_k \text{ are known}$$

The above $f(\theta_1, \dots, \theta_k)$ is called the Dirichlet prior distribution. Denoted as Dirichlet$(\alpha_1, \dots, \alpha_k)$.

then the posterior of θ is

$$f(\theta \mid x_1, \dots, x_k) \propto f(x_1, \dots, x_k \mid \theta)\, f(\theta)$$

$$= \binom{x_1 + \cdots + x_k}{x_1 \ x_2 \ \cdots \ x_k} \theta_1^{x_1} \cdots \theta_k^{x_k} \cdot \frac{\Gamma(\alpha_1 + \cdots + \alpha_k)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_k)} \theta_1^{\alpha_1 - 1} \cdots \theta_k^{\alpha_k - 1}$$

$$\propto \theta_1^{x_1 + \alpha_1 - 1} \cdots \theta_k^{x_k + \alpha_k - 1} \sim \text{Dirichlet}(x_1 + \alpha_1, \dots, x_k + \alpha_k)$$

more about ~~normal~~ normal distribution:

Theorem: If $X \sim N(\mu, \sigma^2)$, then $\frac{X - \mu}{\sigma} \sim N(0, 1)$.

proof: let $Y = \frac{X - \mu}{\sigma}$, then the moment generating function of Y is

$$\psi_Y(t) = E(\exp(tY)) = E\left(\exp\left(\frac{tX}{\sigma} - \frac{t\mu}{\sigma}\right)\right) = e^{-\frac{t\mu}{\sigma}} \cdot E\left(\exp\left(\frac{t}{\sigma} X\right)\right)$$

$$= e^{-\frac{t\mu}{\sigma}} \exp\left(\frac{t}{\sigma}\mu + \frac{1}{2}\left(\frac{t}{\sigma}\right)^2 \cdot \sigma^2\right) = \exp\left(\frac{1}{2}t^2\right).$$

Eg 13. If $X \sim N(1, 4)$. Find $p(X < 2)$.

Sol: $p(X < 2) = p\left(\frac{X-1}{2} < 0.5\right) = p(Z < 0.5) = 1 - \Phi(-0.5)$

$= 1 - 0.3085 = 0.6915$

Eg 14. If in Eg 8. $\bar{X} = 0$, $n = 100$, $\mu_0 = 0$. $\sigma_0^2 = \tau_0^2 = 1$.

Find $p(\mu < 0.01 \mid s)$.

Sol: $\mu \mid s \sim N(0, \frac{1}{101})$. $\boxed{\text{So. } p(10.04988\,\mu < 0.1 \mid s)}$

$p(\mu < 0.01 \mid s) = p(10.04988\,\mu < 0.1 \mid s) = \Phi(0.1) = 1 - \Phi(-0.1)$

$= 1 - 0.4602 = 0.5398$