STAT 417 Lecture NOTE 8

E.g. 4.

Multinomial distribution: Suppose $X_1, X_2, \dots, X_k$ are ~~integer valued~~ random variables such that $X_1 + X_2 + \dots + X_k = n$, where n is a deterministic integer.

we say that $(X_1, X_2, \dots, X_k)$ follows multinomial distribution, with parameters $\theta = (\theta_1, \theta_2, \dots, \theta_k)$. if the ~~probability~~ distribution function of $(X_1, X_2, \dots, X_k)$ is

$$f_\theta(x_1, \dots, x_k) = \binom{n}{x_1\ x_2 \dots x_k} \theta_1^{x_1} \theta_2^{x_2} \dots \theta_k^{x_k}.$$

E.g. when $k=3$. $x_1 = 2$. $x_2 = 3$, $x_3 = 4$, $n = 9$.

then $f_\theta(x_1, x_2, x_3) = \binom{9}{2\ 3\ 4} \theta_1^2 \theta_2^3 \theta_3^4$.

multinomial model and likelihood function. parameter space is

$$\Theta = \{(\theta_1, \dots, \theta_k) \mid 0 \le \theta_1, \dots, \theta_k \le 1, \quad \theta_1 + \dots + \theta_k = 1\}.$$

Suppose samples are $(X_1, \dots, X_k) \sim \text{multinomial}(n; \theta_1, \theta_2, \dots, \theta_k)$.

Then the likelihood function is

$$L(\theta \mid x_1, x_2, \dots, x_k) = f_\theta(x_1, x_2, \dots, x_k) = \binom{n}{x_1 \dots x_k} \theta_1^{x_1} \theta_2^{x_2} \dots \theta_k^{x_k}.$$

So $l(\theta \mid x_1, \dots, x_k) = \log L(\theta \mid x_1, \dots, x_k) = \log \binom{n}{x_1 \dots x_k} + x_1 \log \theta_1 + \dots + x_k \log \theta_k.$

$$= \log \binom{n}{x_1 \dots x_k} + x_1 \log \theta_1 + \dots + x_{k-1} \log \theta_{k-1} + x_k \log(1 - \theta_1 - \theta_2 - \dots - \theta_{k-1})$$

So $$\frac{\partial}{\partial\theta_1} l(\theta \mid x_1 \cdots x_k) = \frac{x_1}{\theta_1} - \frac{x_k}{1-\theta_1-\theta_2-\cdots-\theta_{k-1}} = 0$$

$$\frac{\partial}{\partial\theta_2} l(\theta \mid x_1 \cdots x_k) = \frac{x_2}{\theta_2} - \frac{x_k}{1-\theta_1-\theta_2-\cdots-\theta_{k-1}} = 0$$

$$\vdots$$

$$\frac{\partial}{\partial\theta_{k-1}} l(\theta \mid x_1 \cdots x_k) = \frac{x_{k-1}}{\theta_{k-1}} - \frac{x_k}{1-\theta_1-\theta_2-\cdots-\theta_{k-1}} = 0.$$

solving these equations. we get $\frac{x_1}{\theta_1} = \frac{x_2}{\theta_2} = \cdots = \frac{x_{k-1}}{\theta_{k-1}} = \frac{x_k}{1-\theta_1-\theta_2-\cdots-\theta_{k-1}}$

$$\theta_1 = \frac{x_1}{n}, \quad \theta_2 = \frac{x_2}{n}, \quad \cdots, \quad \theta_k = \frac{x_k}{n}.$$

so $\hat{\theta}_{ML} = \left(\frac{x_1}{n}, \frac{x_2}{n}, \cdots, \frac{x_k}{n} \right).$

E.g. 5. If X_1, X_2 are iid samples from $unif(0, \theta)$.

Find ① $L(\theta \mid x_1, x_2)$, ② $\hat{\theta}_{ML}$.

① $L(\theta \mid x_1, x_2) = f_\theta(x_1, x_2) = f_\theta(x_1) f_\theta(x_2).$

recall for $unif(0, \theta)$, the pdf is

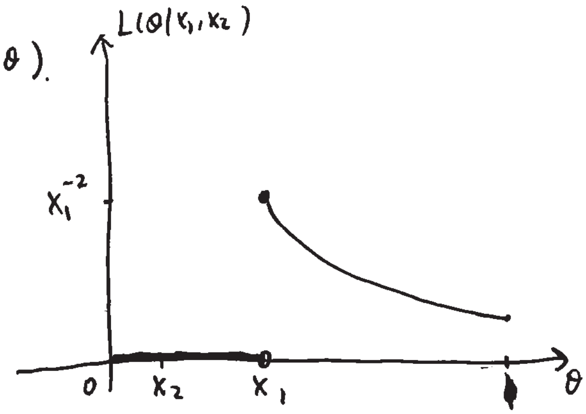
$$f_\theta(x) = \begin{cases} \frac{1}{\theta}, & \text{if } 0 \le x \le \theta, \\ 0, & o/w. \end{cases}$$

using indicator, we rewrite $f_\theta(x) = \frac{1}{\theta} \mathbb{1}(0 \le x \le \theta).$

suppose $x_1 > x_2$.

So $$L(\theta \mid x_1, x_2) = \frac{1}{\theta^2} \mathbb{1}(0 \le x_1 \le \theta) \cdot \mathbb{1}(0 \le x_2 \le \theta).$$

$$= \frac{1}{\theta^2} \mathbb{1}(\theta \ge \max\{x_1, x_2\}).$$



$\hat{\theta}_{ML} = \max\{x_1, x_2\}.$

plug-in ~~MLE~~ MLE estimation.

If we want to estimate $\psi = \gamma(\theta)$, a function of θ.
and we have obtained $\hat{\theta}_{ML}$, then a ~~plug-in [illegible]~~
natural estimation of ψ is

$$\hat{\psi} = \gamma(\hat{\theta}_{ML}).$$

$\hat{\psi}$ is called a plug-in ~~MLE~~ estimation MLE of ψ.

<u>E.g. 6.</u> For $X_1, X_2, \ldots, X_n$ iid from $N(\mu, \sigma^2)$.

$$\hat{\mu}_{MLE} = \bar{X}, \quad \hat{\sigma}^2_{ML} = \frac{n-1}{n} s^2.$$

Then the plug-in estimation MLE of e^{μ} is $e^{\bar{X}}$.

the plug-in MLE of σ is $\sqrt{\frac{n-1}{n} s^2}$.