STAT 417
Lecture Note 16

Statistical inference has three components:

1. estimation.
2. confidence interval.
3. hypothesis testing.

In Bayesian inference, we also have three components:

1. Bayesian estimation.
2. Bayesian confidence interval (or credible interval)
3. Bayesian hypothesis testing.

Note: Bayesian inference is based on posterior distribution.

§7.2.1. Bayesian estimation.

θ is a parameter. we are interested in $\psi = \psi(\theta)$.

Question: How to estimate $\psi(\theta)$?

Solution i: Find MLE of θ, i.e. $\hat{\theta}_{ML}$, then estimate $\psi = \psi(\theta)$ by plug-in MLE, $\hat{\psi} = \psi(\hat{\theta}_{ML})$

Solution ii: Bayesian method estimates ψ by the posterior mean. or posterior mode posterior mean: Suppose the posterior distribution of θ is $f(\theta | s)$, where s is the sample, then the posterior mean of $\psi = \psi(\theta)$ is

$$E(\psi(\theta) | s).$$

In Bayesian method, we estimate ψ by $E(\psi(\theta) | s)$. i.e.

$$\hat{\psi} = E(\psi(\theta) | s).$$

posterior variance of ψ is $Var(\psi(\theta) | s) = E(\psi(\theta)^2 | s) - [E(\psi(\theta) | s)]^2$.

<u>Eg 1</u>. (Normal model) If $X_1, X_2, \ldots, X_n$ are iid samples from $N(\mu, \sigma_0^2)$, where σ_0^2 is known, μ is unknown. The prior for μ is $N(\mu_0, \tau_0^2)$. so the posterior distribution for μ is

$$\mu | s \sim N\left(\frac{n\bar{x}\tau_0^2 + \mu_0\sigma_0^2}{n\tau_0^2 + \sigma_0^2}, \frac{\sigma_0^2\tau_0^2}{n\tau_0^2 + \sigma_0^2}\right).$$

the posterior mean of μ is

$$E(\mu | x_1, \ldots, x_n) = \frac{n\bar{x}\tau_0^2 + \mu_0\sigma_0^2}{n\tau_0^2 + \sigma_0^2}.$$

The posterior variance of μ is

$$Var(\mu | X_1, \ldots, X_n) = \frac{\sigma_0^2\tau_0^2}{n\tau_0^2 + \sigma_0^2}.$$

Eg 2.

(Bernoulli model). Let $X_1, \ldots, X_n$ be i.i.d samples from

$$P(X_i = 1) = \theta, \quad 0 \le \theta \le 1.$$
$$P(X_i = 0) = 1 - \theta.$$

Consider prior on θ as $Beta(\alpha, \beta)$, $\alpha > 0, \beta > 0$ are known.

From Eg 7 in previous ~~chap~~ section. (§7.1 ?).

the posterior distribution of θ is also Beta, more specifically,

$$\theta \mid X_1, \ldots, X_n \sim Beta(n\bar{x} + \alpha, \; n(1-\bar{x}) + \beta).$$

so the posterior mean of θ is

$$E(\theta \mid X_1, \ldots, X_n) = \frac{n\bar{x} + \alpha}{n + \alpha + \beta}.$$

the posterior variance of θ is

$$Var(\theta \mid X_1, \ldots, X_n) = \frac{(n\bar{x} + \alpha)(n(1-\bar{x}) + \beta)}{(n + \alpha + \beta + 1)(n + \alpha + \beta)^2}.$$

Eg3. If X is a sample from Exponential(θ), for unknown $\theta > 0$.

θ has prior Exponential(θ_0), for some known $\theta_0 > 0$.

Find the posterior mean and posterior variance of θ.

sol: $f(\theta | x) \propto f(x | \theta) f(\theta)$

$$= \theta e^{-\theta x} \cdot \theta_0 e^{-\theta_0 \theta}.$$

$$= \theta_0 \theta e^{-(x+\theta_0)\theta}.$$

so $\theta | x \sim$ Gamma$(2, x+\theta_0)$.

then the posterior mean of θ is

$$E(\theta | x) = \frac{2}{x+\theta_0}.$$

the posterior variance of θ is

$$Var(\theta | x) = \frac{2}{(x+\theta_0)^2}.$$

EX1. If $X_1, X_2, \cdots, X_n$ are iid samples from Exponential (θ). for some unknown $\theta > 0$. θ has prior Exponential (θ_0). for some known $\theta_0 > 0$. Find the posterior mean and variance of θ.

sol: $f(\theta \mid x_1, \cdots, x_n) \propto f(x_1, \cdots, x_n \mid \theta) f(\theta)$

$$= \theta e^{-x_1 \theta} \cdots \theta e^{-x_n \theta} \cdot \theta_0 e^{-\theta_0 \theta}$$

$$= \theta_0 \theta^n e^{-(x_1 + \cdots + x_n + \theta_0)\theta}$$

$$= \theta_0 \theta^n e^{-(n\bar{x} + \theta_0)\theta}.$$

so. θ has posterior distribution

$$\theta \mid x_1, \cdots, x_n \sim \text{Gamma}(n+1, \ n\bar{x} + \theta_0).$$

so the posterior mean of θ is

$$E(\theta \mid X_1, \cdots, X_n) = \frac{n+1}{n\bar{x} + \theta_0}.$$

the posterior variance of θ is

$$\text{Var}(\theta \mid x_1, \cdots, x_n) = \frac{n+1}{(n\bar{x} + \theta_0)^2}.$$

Eg 4. In Eg 3. Find the posterior mean ~~and variance~~ of $\frac{1}{\theta}$.

Sol: posterior mean of $\frac{1}{\theta}$ is

$$E\left(\frac{1}{\theta} \mid x\right) = \int_0^\infty \frac{1}{\theta} f(\theta \mid x)\, d\theta = \int_0^\infty \frac{1}{\theta} \cdot \frac{(x+\theta_0)^2}{\Gamma(2)} \theta e^{-(x+\theta_0)\theta} d\theta$$

$$= \frac{(x+\theta_0)^2}{\Gamma(2)} \int_0^\infty e^{-(x+\theta_0)\theta} d\theta$$

$$= \frac{(x+\theta_0)}{\Gamma(2)} = x+\theta_0, \quad (\text{note } \Gamma(2) = 1)$$

EX2. Suppose the posterior distribution of θ given sample s is

$$f(\theta \mid s) = \begin{cases} e^{-(\theta - s)}, & \text{if } \theta \geq s \\ 0, & \text{o/w} \end{cases}$$

Find the posterior mean of θ.

Sol: $$E(\theta \mid s) = \int_s^\infty \theta e^{-(\theta - s)} d\theta$$

$$= \int_0^\infty (\theta + s) e^{-\theta} d\theta = \int_0^\infty \theta e^{-\theta} d\theta + s \int_0^\infty e^{-\theta} d\theta$$

$$= 1 + s.$$