STAT 413 lecture Note 19

Suppose the ~~th~~ posterior distribution of θ given the sample S is $f(\theta|S)$. For any ~~real~~ set of real numbers A, we have

$$P(\theta \in A | S) = \int_A f(\theta|S)\, d\theta.$$ (This the STAT 416 material)

The posterior odds ratio in favor of $H_0: \theta \in A$ is

$$\frac{P(H_0|S)}{P(H_1|S)} = \frac{P(\theta \in A|S)}{P(\theta \notin A|S)} = \frac{P(\theta \in A|S)}{1 - P(\theta \in A|S)}$$

If we change the posterior distribution to be prior distribution. $f(\theta)$. we can similarly define the so-called <u>prior odds ratio in favor of H_0.</u>

$$\frac{P(H_0)}{P(H_1)} = \frac{P(\theta \in A)}{P(\theta \notin A)} = \frac{P(\theta \in A)}{1 - P(\theta \in A)}.$$

where $P(\theta \in A)$ is the prior probability that $\theta \in A$, or, H_0 is true. we use $f(\theta)$ to calculate $P(\theta \in A)$, i.e.,

$$P(\theta \in A) = \int_A f(\theta)\, d\theta,$$ where $f(\theta)$ is the prior of θ.

We could use prior distribution to test H_0 vs H_1 by the following rule:

if $\frac{P(H_0)}{P(H_1)} < 1$, then reject H_0;

if $\frac{P(H_0)}{P(H_1)} > 1$, the do not reject H_1;

Eg 3. Suppose we want to test $H_0: \theta \leq 1$. vs $H_1: \theta > 1$.

the prior distribution on θ is $N(0,1)$. ① Find the prior odds ratio in favor of H_0, and ② Based on ①, make decision on whether or not you will reject H_0?

Sol: $p(H_0) = P(\theta \leq 1) = \Phi(1) = 1 - \Phi(-1) = 1 - 0.1587 = 0.8413$

so. the prior odds ratio is

$$\frac{p(H_0)}{1-p(H_0)} = \frac{0.8413}{1-0.8413} = 5.30 > 1.$$

so. we do not reject H_0.

Eg 4. Suppose We want to test $H_0: 0 \leq \theta \leq 1$. vs $H_1: \theta > 1$.

The prior on θ is $\theta \sim$ Exponential (1). ① Find the prior odds ratio in favor of H_0 ②. do you reject H_0?

Sol: $p(H_0) = \int_0^1 e^{-\theta} d\theta = 0.63$

so the prior odds ratio in favor of H_0 is

$$\frac{p(H_0)}{1-p(H_0)} = \frac{0.63}{1-0.63} = 1.72 > 1.$$

so. we don't reject H_0.

EX1. Let θ = lifetime of a Garmin GPS. Our prior distribution on θ is

$$f(\theta) = \begin{cases} \dfrac{1}{(1+\theta)^2}, & \text{if } \theta \geq 0 \\ 0. & \text{o/w.} \end{cases}$$

Test $H_0: 1 \leq \theta \leq 2$ by prior odds ratio in favor of H_0.

Sol: $$p(H_0) = \int_1^2 \frac{1}{(1+\theta)^2}\, d\theta = -\frac{1}{1+\theta}\Big|_1^2 = \frac{1}{6}.$$

So $$\frac{p(H_0)}{1-p(H_0)} = \frac{\frac{1}{6}}{1-\frac{1}{6}} = 0.2 < 1.$$

so we reject H_0.

EX2: Let θ = lifetime of Garmin GPS. the prior on θ is Exponential(α), for $\alpha > 0$. Test H_0: $\theta \geq 1$. by prior odds ratio. in favor of H_0.

For what α do you reject H_0?

Sol: $\cancel{p(H_0) = \int_1^2 \alpha e^{-\alpha\theta}\, d\theta = -e^{-2\alpha} + e^{-\alpha}}$.

$$p(H_0) = \int_1^\infty \alpha e^{-\alpha\theta}\, d\theta = e^{-\alpha}.$$

The prior odds ratio $= \dfrac{e^{-\alpha}}{1-e^{-\alpha}}$. reject H_0 if $\dfrac{e^{-\alpha}}{1-e^{-\alpha}} < 1$.

so $\alpha > \log 2 \approx 0.693$.

STAT 417 lecture note 20

Bayes Factor.

Bayes factor is the third way for Bayesian hypothesis testing.

To understand the basic principal, look at a simple example.

<u>Eg1.</u> Suppose we toss a coin $n=10$ times. ~~and~~ each time,

$P(\text{get } H) = p$, $P(\text{get } T) = 1-p$. but we do not know the value of p. If p has two candidate values, $p=0.5$ or $p=0.9$.

($p=0.5$ corresponds to fair coin, $p=0.9$ corresponds unfair coin).

We want to decide which value is correct, i.e. we test

$$H_0: p=0.5 \quad \text{vs} \quad H_1: p=0.9.$$

After the 10 tosses, we observe $\underbrace{X=6}_{\text{let } X=\#\text{ Hs among the 10 tosses}}$ Heads ~~and~~

~~which~~ Question: what is our decision?

Bayes solves the problem based on the following thinking:

If H_0 is correct, i.e., $p=0.5$, then we calculate the probability of getting $X=6$ heads. Suppose the probability is very small, which will contradict our hypothesis H_0, then we believe that H_0 is incorrect.

But if the probability of getting $X=6$ heads, is big, then we believe that H_0 is plausible

The flow chart shows the idea:

$$\begin{array}{ccc} \boxed{H_0 :\ p=0.5} & & \boxed{H_1 :\ p=0.9} \\ \downarrow & & \downarrow \\ p(X=6) & & p(X=6) \\ \text{or} & & \text{or} \\ p(X=6 \mid H_0) & & p(X=6 \mid H_1) \end{array}$$

If $\quad p(X=6 \mid H_0) > p(X=6 \mid H_1)$. Then prefer H_0.

If $\quad p(X=6 \mid H_0) < p(X=6 \mid H_1)$. Then prefer H_1.

Next we exactly calculate the two probabilities.

If H_0 is correct. $p=0.5$, then $X \sim \text{Binomial}(10, 0.5)$.

so $\quad p(X=6 \mid H_0) = \binom{10}{6} 0.5^6 (1-0.5)^{10-6} = (210)(0.5)^{10} = 0.21$.

If H_1 is correct. $p=0.9$. then $X \sim \text{Binomial}(10, 0.9)$.

so $\quad p(X=6 \mid H_1) = \binom{10}{6} 0.9^6 (1-0.9)^{10-6} = (210)(0.9^6)(0.1)^4 = 0.01$

So. $\quad p(X=6 \mid H_0) > p(X=6 \mid H_1)$. which means that H_0 is NOT rejected.

The Bayes factor is defined to be

$$\frac{p(X=6 \mid H_0)}{p(X=6 \mid H_1)} = \frac{0.21}{0.01} = 21. > 1. \quad \text{so } H_0 \text{ is rejected.}$$

In general, Suppose we want to test H_0 vs H_1, and we have observation s, then the Bayes factor ~~is defined~~ in favor of H_0 is defined as

$$BF_{H_0} = \frac{p(s \mid H_0)}{p(s \mid H_1)}$$

If $BF_{H_0} > 1$. then do not reject H_0.

$BF_{H_0} < 1$. then reject H_0.

Eg2. Suppose ~~the number of~~ suicides occur in a population at a rate p per person per year. p is unknown, If we model the number of suicides in a population of total N persons as poisson (Np). and we observed 22 suicides with $N = 30345$.

Test . by Bayes factor, $H_0: p = 0.001$ vs $H_1: p = 0.0001$

Sol: let X = # suicides in the N people.

so $X \sim$ poisson (Np)

so. $$P(X=22 \mid H_0) = \frac{[(30345)(0.001)]^{22}}{22!} e^{-(30345)(0.001)} = 0.024$$

$$P(X=22 \mid H_1) = \frac{[(30345)(0.0001)]^{22}}{22!} e^{-(30345)(0.0001)} = 1.73 \times 10^{-12}.$$

so. $$BF_{H_0} = \frac{0.024}{1.73 \times 10^{-12}} \doteq 1.38 \times 10^{8}. > 1.$$ So. Do not reject H_0.

Relationship between Bayes factor and prior / posterior odds ratio.

<u>theorem:</u> For testing H_0 vs H_1.

$$BF_{H_0} = \frac{\text{posterior odds ratio in favor of } H_0}{\text{prior odds ratio in favor of } H_0}$$

<u>proof</u>: $$BF_{H_0} = \frac{p(s \mid H_0)}{p(s \mid H_1)} = \frac{p(s, H_0) / p(H_0)}{p(s, H_1) / p(H_1)}$$

$$= \frac{p(s, H_0)}{p(s, H_1)} \Big/ \frac{p(H_0)}{p(H_1)}$$

$$= \left(\frac{p(s, H_0) / p(s)}{p(s, H_1) / p(s)} \right) \Big/ \left(\frac{p(H_0)}{p(H_1)} \right)$$

$$= \frac{p(H_0 \mid s)}{p(H_1 \mid s)} \Big/ \frac{p(H_0)}{p(H_1)}$$

$$= \frac{\text{posterior odds ratio in favor of } H_0}{\text{prior odds ratio in favor of } H_0}.$$

EX1. If X is uniform on $\{1, 2, \cdots, N\}$. where N is unknown integer-valued parameter. Want to test $H_0: N=10$ vs $H_1: N=20$. Suppose we observe a sample of X, which is $X=9$.

Find BF_{H_0} and make decision.

Sol: $p(X=9 \mid H_0) = \frac{1}{10}$, $p(X=9 \mid H_1) = \frac{1}{20}$.

So. $BF_{H_0} = \frac{\frac{1}{10}}{\frac{1}{20}} = 2 > 1$. So Do not reject H_0.

Eg3. ~~If~~ A measurement X follows $N(\mu, 4)$. Want to test $H_0: \mu \le 1$, vs $H_1: \mu > 1$. If we have observed a sample of X, which is $X = 0.9$. ~~Find the Bayes factor~~ If the prior on μ is $\mu \sim N(0, 1)$ Find the BF_{H_0} and make decision.

Sol: $\hat{\mu} = \frac{n\bar{x}\tau_0^2 + \mu_0\sigma_0^2}{n\tau_0^2 + \sigma_0^2} = \frac{0.36}{5} = 0.072$.

$\sigma^2 = \frac{\tau_0^2\sigma_0^2}{n\tau_0^2 + \sigma_0^2} = \frac{(1)(4)}{(1)(1)+4} = \frac{4}{5} = 0.8$

So, the posterior of μ is $N(0.072, 0.8)$.

The posterior odds ratio in favor of H_0 is $\frac{p(\mu \le 1 \mid x)}{p(\mu > 1 \mid x)} = \frac{\Phi(1.038)}{1-\Phi(1.038)} = \frac{0.85}{1-0.85} = 5.7$

The prior odds ratio in favor of H_0 is $\frac{p(\mu \le 1)}{p(\mu > 1)} = \frac{\Phi(1)}{1-\Phi(1)} = \frac{0.84}{1-0.84} = 5.3$.

So $BF_{H_0} = \frac{5.7}{5.3} = 1.08 > 1$. So Do not reject H_0.

Ex2. If $x_1, x_2, \dots, x_{99}$ are iid normal samples from $N(\mu, 1)$.

the prior on μ is standard normal. If it is observed that $\bar{x} = 0.1$

Test $H_0: \mu \le$ [redacted] 0 vs $H_1: \mu >$ [redacted] 0

Sol: $$\hat{\mu} = \frac{(99)(0.1)(1) + (0)(1)}{(99)(1) + 1} = 0.099.$$

$$\sigma^2 = \frac{(1)(1)}{(99)(1)+1} = 0.01$$

. The posterior of μ is $\mu \sim N(0.099, 0.01)$

So. the posterior OR in favor of H_0 is

$$\frac{p(\mu \le 0\ \text{[redacted]} \mid x_1, \dots, x_{99})}{p(\mu > 0\ \text{[redacted]} \mid x_1, \dots, x_{99})} = \frac{\Phi(-0.99)}{1 - \Phi(-0.99)} = \frac{0.16}{1 - 0.16} = 0.19$$

the prior OR in favor of H_0 is

$$\frac{p(\mu \le 0)}{p(\mu > 0)} = \frac{\Phi(0)}{1 - \Phi(0)} = 1.$$

So $BF_{H_0} = \frac{0.19}{1} = 0.19 < 1.$ So reject H_0.

STAT 417 Lecture Note 21.

Eg 4*. If $X_1, X_2, \dots, X_{10}$ are i.i.d samples from uniform $[0, \theta]$ for some unknown parameter $\theta > 0$. Assume a uniform prior on θ as $\theta \sim$ uniform $[0, 1]$. Suppose in practice we observed the samples as

0.1, 0.2, 0.3, 0.4, 0.2, 0.7, 0.85, 0.17, 0.29, 0.31.

Test $H_0: \theta \leq 0.9$ vs $H_1: \theta > 0.9$.

Sol: First of all, find the posterior of θ as

$$f(\theta | x_1, \dots, x_{10}) \propto f(x_1, \dots, x_{10} | \theta) f(\theta)$$

$$= \prod_{i=1}^{10} \theta^{-1} \mathbb{1}_{[0,\theta]}(x_i) \cdot f(\theta)$$

$$= \begin{cases} \theta^{-10}. & \text{if } x_{(10)} \leq \theta \leq 1, \\ 0. & \text{o/w.} \end{cases}$$

where $x_{(10)} = \max\{x_1, \dots, x_{10}\} = 0.85$. That is,

$$f(\theta | x_1, \dots, x_{10}) \propto \theta^{-10}. \quad \text{if } 0.85 \leq \theta \leq 1.$$

Assume that $f(\theta | x_1, \dots, x_{10}) = c\,\theta^{-10}$, $0.85 \leq \theta \leq 1$, where c is constant such that $\int_{0.85}^{1} c\,\theta^{-10} d\theta = 1$, so $c = 2.71$ (not necessarily to find c).

Second. the prior odds ratio in favor of H_0 is

$$\frac{p(H_0)}{p(H_1)} = \frac{\int_{0.}^{0.9} d\theta}{\int_{0.9}^{1} d\theta} = 9.$$

third, the posterior odds ratio in favor of H_0 is

$$\frac{p(H_0 | x_1, \ldots, x_{10})}{p(H_1 | x_1, \ldots, x_{10})} = \frac{\int_{0.85}^{0.9} 2.71\ \theta^{-10}\, d\theta}{\int_{0.9}^{1} 2.71\ \theta^{-10}\, d\theta} = 1.1 > 1.$$

So. the Bayes factor in favor of H_0 is

$$BF_{H_0} = \frac{1.1}{9} = 0.122 < 1.$$ so H_0 is rejected.

EX3. Find the range of Bayes Factor.

Sol:
$$BF_{H_0} = \frac{\text{posterior OR}}{\text{prior OR}}.$$

If posterior OR $= 0$. then $BF_{H_0} = 0$. otherwise $BF_{H_0} > 0$.

So. BF_{H_0} ranges from 0 to ∞. i.e. $BF_{H_0} \in [0, \infty)$.

EX4. If $X_1, X_2, \cdots, X_n$ are i.i.d samples from Exponential(θ). $\theta > 0$ is unknown. let θ have prior as Exponential(1). $n = 10$, $\bar{X} = 2$. use Bayes Factor to test $H_0: \theta \geq 1$ vs $H_1: \theta < 1$.

Hint:
$$\frac{\int_1^\infty \theta^{10} e^{-21\theta} d\theta}{\int_0^1 \theta^{10} e^{-21\theta} d\theta} \doteq 0.006$$

Sol: The posterior for θ is

$$f(\theta | x_1, \cdots, x_n) \propto \prod_{i=1}^n \theta e^{-\theta x_i} \cdot e^{-\theta} = \theta^n e^{-(n\bar{x}+1)\theta}. \quad \theta \geq 0$$

then the prior odds ratio is

$$\frac{P(\theta \geq 1)}{P(\theta < 1)} = \frac{\int_1^\infty e^{-\theta} d\theta}{\int_0^1 e^{-\theta} d\theta} = \frac{e^{-1}}{1 - e^{-1}} = \frac{1}{e-1} = 0.582.$$

the posterior odds ratio is

$$\frac{P(\theta \geq 1 | x_1 \cdots x_n)}{1 - P(\theta \geq 1 | x_1 \cdots x_n)} = \frac{\int_1^\infty \theta^{10} e^{-21\theta} d\theta}{\int_0^1 \theta^{10} e^{-21\theta} d\theta} = 0.006.$$ So $BF_{H_0} = \frac{0.006}{0.582} = 0.01 < 1$.

So. H_0 is rejected.