STAT 417 Lectures Notes 18.

§ 7.2.3. Bayes Factors

A motivating example Eg1: Suppose we have $n = 99$ iid measurements.

$X_1, X_2, \ldots, X_{99}$ from $N(\mu, 1)$, where $\mu \in \mathbb{R}$ is unknown.

Want to test $H_0: \mu \le 0.1$ vs $H_1: \mu > 0.1$

In STAT 350, we talked about $H_0: \mu = \mu_0$. Here, the above H_0 is more challenging. Below is how we solve the problem.

Assume prior distribution on μ: $\mu \sim N(0, 1)$.

Then the posterior of μ is

$$\mu \mid x_1, \ldots, x_{99} \sim N(\hat{\mu}, \sigma^2), \quad \text{with } \hat{\mu} = \frac{99\bar{x} + (0)(1)}{(99)(1) + 1} = \frac{99\bar{x}}{100}$$

$$\sigma^2 = \frac{(1)(1)}{(99)(1) + 1} = \frac{1}{100}.$$

Suppose in practice we observe $\bar{x} = 0$.

Then $\hat{\mu} = 0$. So $\mu \mid x_1, \ldots, x_{99} \sim N(0, 0.01)$

To test H_0 vs H_1, we just evaluate the posterior probabilities of H_0 and H_1: the bigger one is preferred.

Exactly, $p(H_0 \mid x_1, \dots, x_{99}) = p(\mu \le 1 \mid x_1, \dots, x_{99})$

$$= p\left(\frac{\mu - 0}{0.1} \le \frac{0.1 - 0}{0.1} \,\middle|\, x_1, \dots, x_{99} \right)$$

$$= \Phi(1)$$

$$= 1 - \Phi(-1) = 1 - 0.1587 = 0.8413.$$

so, $p(H_1 \mid x_1, \dots, x_{99}) = 1 - 0.8413 = 0.1587.$

so $p(H_0 \mid x_1, \dots, x_{99}) > p(H_1 \mid x_1, \dots, x_{99}).$

so we do NOT reject H_0.

The above procedure makes much sense since our decision is made based on the probability of H_0 and H_1. The bigger one is of course preferred.

The ratio $\dfrac{p(\mu \le 0.1 \mid x_1, \dots, x_{99})}{p(\mu > 0.1 \mid x_1, \dots, x_{99})}$ is called ~~Bayes factor~~ the posterior odds ratio in favor of H_0.

we always compare ~~Bayes Factor~~ posterior odds ratio with one to make decision.

More generally, if A is a set of $\mathbb{R}$. we want to test

H_0: $\theta \in A$

H_1: $\theta \notin A$.

Suppose we have got the posterior distribution of θ, which is $f(\theta | s)$, where s is the sample. then the posterior odds ratio in favor of H_0 is

$$\frac{P(H_0|s)}{p(H_1|s)} = \frac{p(\theta \in A \mid s)}{p(\theta \notin \mid s)}$$

If the ratio is greater than one, then we prefer H_0, otherwise, we prefer H_1. Note that the above ratio is equal to

$$\frac{p(\theta \in A \mid s)}{1 - p(\theta \in A \mid s)}.$$

<u>Eg 2.</u> Suppose the posterior distribution of θ is $f(\theta | s)$ defined as

$$f(\theta|s) = \begin{cases} e^{-(\theta-s)}, & \text{if } \theta \geq s, \\ 0 & \text{if } \theta < s \end{cases}$$

Test H_0: $\theta > 2$. vs H_1: $\theta \leq 2$.

① if $s = 0.9$

② if $s = 1$.

Sol: ① $p(\theta \geq 2 \mid s) = \int_2^\infty e^{-(\theta - 1.9)}\, d\theta = \int_{0.1}^\infty e^{-\theta}\, d\theta = e^{-0.1} = 0.9048.$

$p(\theta \leq 2 \mid s) = 1 - 0.9048 = 0.0952$

so the posterior odds ratio in favor of H_0 is

$$\frac{0.9048}{0.0952} = 9.51 > 1.$$

so we prefer H_0, i.e., do not reject H_0.

②. $p(\theta > 2 \mid s) = \int_2^\infty e^{-(\theta-1)}\, d\theta = \int_1^\infty e^{-\theta}\, d\theta = e^{-1} = 0.3679.$

so $p(\theta \leq 2 \mid s) = 1 - 0.3679 = 0.6321.$

so the posterior odds ratio of ~~th~~ favor of H_0 is

$$\frac{0.3679}{0.6321} = 0.58 < 1.$$

so we reject H_0.

Ex1. For what value of the sample s do you reject H_0? In Eg 2.

Sol: If $s > 2$. then $p(H_0 \mid s) = \int_2^\infty f(\theta \mid s)\, d\theta = \int_s^\infty e^{-(\theta - s)}\, d\theta = 1.$

So the posterior odds ratio in favor of H_0 is $\frac{1}{1-1} = \infty > 1.$

so we always do not reject H_0.

If $0 \leq s \leq 2$, $p(H_0 \mid s) = \int_2^\infty e^{-(\theta - s)}\, d\theta = \int_{2-s}^\infty e^{-\theta}\, d\theta = e^{-(2-s)}$.

the posterior odds ratio is $\frac{e^{-(2-s)}}{1 - e^{-(2-s)}} < 1$ to reject H_0.

so $s < \log\left(e^{2}/\right.$[illegible]$\left.\right) \approx 1.31$