HW 2. Solutions

1. a). $$\int_{x_0}^{+\infty} f(x|X>x_0)\,dx = \frac{\int_{x_0}^{+\infty} f(x)\,dx}{1-F(x_0)} = 1.$$

$f(x|X>x_0)$ is also positive. So it is a pdf.

b) $$F(x) = \int_0^x e^{-y}\,dy = 1-e^{-x}.$$

$$f(x|X>x_0) = \frac{e^{-x}}{e^{-x_0}} = e^{x_0-x}.$$

$$P(X>2|X>1) = \int_2^{+\infty} e^{1-x}\,dx = e^{-1}.$$

2. $$h(v) = E\left(\left((X-\mu_1)+v(Y-\mu_2)\right)^2\right) = E\left((X-\mu_1)^2+v^2(Y-\mu_2)^2+2v(X-\mu_1)(Y-\mu_2)\right)$$

$$= \sigma_1^2 + 2v\rho\sigma_1\sigma_2 + v^2\sigma_2^2.$$

Since $h(v)\geq 0$. $4\rho^2\sigma_1^2\sigma_2^2 - 4\sigma_1^2\sigma_2^2 \leq 0$, $|\rho|\leq 1$

3. a). $$M(t_1,t_2) = Ee^{t_1X+t_2Y} = \sum_{y=0}^{\infty}\sum_{x=0}^{y} e^{t_1x+t_2y}\frac{e^{-2}}{x!(y-x)!}$$

$$= \sum_{y=0}^{\infty}\frac{e^{t_2y}e^{-2}}{y!}\sum_{x=0}^{y}\binom{y}{x}e^{t_1x} = \sum_{y=0}^{\infty}\frac{e^{t_2y}e^{-2}}{y!}\cdot(e^{t_1}+1)^y$$

$$= e^{(e^{t_1+t_2}+e^{t_2}-2)}$$

b)

$$\frac{\partial}{\partial t_1} M(t_1,t_2) = e^{e^{t_1+t_2}+e^{t_2}-2} \cdot e^{t_1+t_2}$$

$$\frac{\partial}{\partial t_2} M(t_1,t_2) = e^{e^{t_1+t_2}+e^{t_2}-2} \cdot \left(e^{t_1+t_2}+e^{t_2}\right)$$

$$\frac{\partial^2}{\partial t_1^2} M(t_1,t_2) = e^{e^{t_1+t_2}+e^{t_2}-2} \cdot \left(e^{2(t_1+t_2)} + e^{t_1+t_2}\right)$$

$$\frac{\partial^2}{\partial t_2^2} M(t_1,t_2) = e^{e^{t_1+t_2}+e^{t_2}-2} \left(\left(e^{t_1+t_2}+e^{t_2}\right)^2 + e^{t_1+t_2}+e^{t_2}\right)$$

$$\frac{\partial^2}{\partial t_1 \partial t_2} M(t_1,t_2) = e^{e^{t_1+t_2}+e^{t_2}-2} \left(e^{t_1+t_2}\left(e^{t_1+t_2}+e^{t_2}\right) + e^{t_1+t_2}\right)$$

let $t_1 = t_2 = 0$.

$EX = 1$, $EY = 2$. $EX^2 = 2$ $EY^2 = 6$

$E(XY) = 3$ $Var(X) = 1.$ $Var(Y) = 2$

$cov(X,Y) = 1.$ $\rho = \frac{1}{\sqrt{2}}$

c) $\cdot$ $f(x|y) = \frac{f(x,y)}{f(y)}$

$$f(y) = \sum_{x=0}^{y} \frac{e^{-2}}{x!(y-x)!} = \frac{e^{-2}}{y!} \sum_{x=0}^{y} \binom{y}{x} = \frac{e^{-2}2^y}{y!}$$

$$f(x|y) = \frac{\frac{e^{-2}}{x!(y-x)!}}{\frac{e^{-2}2^y}{y!}} = \binom{y}{x} 2^{-y}$$ $X|Y \sim \text{Binomial}(Y, \frac{1}{2})$

$$E(X|Y=y) = \frac{Y}{2}$$

4. $\begin{cases} Y = \dfrac{X_1}{X_2} \\ W = X_1 \end{cases} \Rightarrow \begin{cases} X_1 = W & 0 < W < 1 \\ X_2 = \dfrac{W}{Y} & 0 < \dfrac{W}{Y} < 1 \end{cases} \quad 0 < W < \min(Y, 1)$

$$J = \begin{vmatrix} \frac{\partial X_1}{\partial Y} & \frac{\partial X_1}{\partial W} \\ \frac{\partial X_2}{\partial Y} & \frac{\partial X_2}{\partial W} \end{vmatrix} = \begin{vmatrix} 0 & 1 \\ -\frac{W}{Y^2} & \frac{1}{Y} \end{vmatrix} = \frac{W}{Y^2}$$

The joint distribution of Y and W is

$$f(Y, W) = \frac{W}{Y^2} \qquad 0 < W < \min(Y, 1)$$

$$f(Y) = \int_0^{\min(Y,1)} \frac{W}{Y^2}\, dy = \begin{cases} \frac{1}{2} & \text{if } Y < 1 \\ \frac{1}{2Y^2} & \text{if } Y \geq 1. \end{cases}$$

$$F(Y) = \begin{cases} \frac{Y}{2} & \text{if } Y < 1 \\ 1 - \frac{1}{2Y} & \text{if } Y \geq 1 \end{cases}$$

5. $$E e^{t(UV)} = \iint e^{tuv} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}\, du dv$$

$$= \frac{1}{2\pi} \iint e^{-\frac{u^2}{2} + tuv - \frac{v^2}{2}}\, du dv \qquad \text{let } \begin{cases} t = \rho \\ \frac{1}{1-t^2} = \sigma^2 \end{cases}$$

$$= \frac{1}{2\pi} \iint e^{-\frac{1}{2(1-\rho^2)}\left[\frac{x^2}{\sigma^2} - 2\rho\frac{xy}{\sigma^2} + \frac{y^2}{\sigma^2}\right]} dx dy \longleftarrow \text{note a bivariate normal pdf.}$$

$$= \frac{1}{2\pi} \cdot 2\pi \sigma^2 \sqrt{1-\rho^2}. = (1-t^2)^{-\frac{1}{2}}.$$