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Robust optimal strategies for an insurer with reinsurance and investment under benchmark and mean-variance criteria

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Abstract In this paper, an ambiguity-averse insurer (AAI) whose surplus process is approximated by a Brownian motion with drift, hopes to manage risk by both investing in a Black-Scholes financial market and transferring some risk to a reinsurer, but worries about uncertainty in model parameters. She chooses to find investment and reinsurance strategies that are robust with respect to this uncertainty, and to optimize her decisions in a mean-variance framework. By the stochastic dynamic programming approach, we derive closed-form expressions for a robust optimal benchmark strategy and its corresponding value function, in the sense of viscosity solutions, which allows us to find a mean-variance efficient strategy and the efficient frontier. Furthermore, economic implications are analyzed via numerical examples. In particular, our conclusion in the mean-variance framework differs qualitatively, for certain parameter ranges, with model-uncertainty robustness conclusions in the framework of utility functions: Model uncertainty does not always result in an agent deciding to reduce risk exposure under mean-variance criteria, opposite to the conclusions for utility functions in Maenhout (2006) and Liu (2010). Our conclusion can be interpreted as saying that the mean-variance problem for the AAI explains certain counter-intuitive investor behaviors, by

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which the attitude to risk exposure, for an AAI facing model uncertainty, depends on positive past experience.

Keywords Stochastic optimization · mean-variance · ambiguity-averse insurer · Black-Scholes · model uncertainty

1 Introduction

The mean-variance portfolio selection theory proposed in Markowitz (1952) is recognized as a cornerstone of modern finance. By providing a clear framework on how to find an optimal allocation strategy among risky assets in order to achieve a given expected return with minimal variance, it has played a significant role both in academia and industry due to its simplicity and practicality, and has inspired literally hundreds of extensions and applications; see Hakasson (1971), Li & Ng (2000), Celikyurt & Özekici (2007), Wu & Li (2012) for discrete-time problems.

For continuous-time markets, Zhou & Li (2000) introduced the linear-quadratic (LQ) stochastic control approach to derive the optimal mean-variance portfolio. Li et al. (2002) obtained the optimal mean-variance portfolio with no-shorting constraints via the stochastic dynamical programming approach: It turns out that the corresponding value function, solution to a Hamilton-Jacobi-Bellman (HJB) equation, only exists in the sense of viscosity solutions. They also illustrate that, generally, under mean-variance criteria, control constraints lead to non-smooth value functions, requiring the interpretation of HJB solution in the sense of viscosity.

Bäuerle (2005) pointed out that mean-variance criteria can also be of interest in insurance applications, and started research on optimal reinsurance problems under benchmark and mean-variance criteria. In the meantime, going beyond the use and continued development of reinsurance as a traditional risk-spreading approach, insurance practitioners and academics have looked at risky investments as a viable and in some cases highly significant way to utilize insurer surplus. Recently, some scholars investigated optimal reinsurance and/or investment problems for insurers under utilities or mean-variance framework, including the use of the Black-Scholes framework for modeling market risk: see among others, Schmidli (2001), Yang & Zhang (2005), Gu et al. (2012) for utilities, Bäuerle (2005), Delong & Gerrard (2007), Zeng et al. (2010) and Zeng & Li (2011) for mean-variance criteria.

In a different direction, some scholars advocated and investigated the impact of economic model uncertainty on portfolio selection. They pointed out that in many cases, the parametric models used in theory, such as the Black-Scholes model, contain significant uncertainties in parameter estimates, particularly in the so-called drift parameters. In practice, this means that the expectation of the return process on a risky asset is not known a priori with any adequate precision, and the investor usually has to account for a significant level of error in drift parameter estimates. For an insurer who considers risky market investments in a Black-Scholes framework, the situation is necessarily identical, and moreover, accurate estimation of the surplus model's drift parameter, i.e. its expectation, can also be called into question. A wise insurer who faces uncertainty in all drift parameters would then hope for a systematic and quantitative way to take such model uncertainty into account.

There are several strategies proposed in the literature to implement portfolio selection under model uncertainty, and one could arguably take any one of these several routes when embarking on the task of adapting these methods to the insurance business. For example, Mataramvura & Øksendal (2008) and Zhang & Siu (2009) investigated model uncertainty for investors/insurers via stochastic differential game theory. In another study, Nietert (2003) disclosed the difficulties caused by model uncertainty in practice: He showed that even (options-based) portfolio insurance can not protect minimum investment goals with probability one, because of model uncertainty on the market price of risk. In this paper, we follow the approach advanced by Anderson et al. (1999), which introduces the concept of ambiguity-aversion and formulates a robust control problem for investors. Uppal & Wang (2003) extended Anderson et al. (1999) by developing a model-uncertainty robustness framework with different levels of ambiguity. Maenhout (2004, 2006) innovated a "homothetic robustness" framework which allowed him to derive explicit closed-form solutions to dynamic robust portfolio optimizations for an investor with constant relative risk aversion (CRRA) utility. Liu (2010) studied the optimal investment and consumption strategy for an investor under "homothetic robustness", and obtained the robust optimal strategy under recursive preferences.

To adapt these general strategies to the insurance domain, recall that our goal is to help our AAI manages her risk by taking advantage of risky Black-Scholes markets to invest her surplus process, while agreeing with her premise that the estimated models (also called the reference model) contain significant parameter uncertainty, i.e. that the market (true models) may deviate from the reference model. We will assume that she has some quantitative preferences regarding model uncertainty, which must be taken into account by considering alternative models which are close to the reference model. We will define a strategy which is robust with respect to (w.r.t.) these alternatives. Very few papers consider reinsurance-and-investment strategies for an insurer with individual preferences when facing model uncertainty, see Yi et al. (2013). In particular, optimizing such strategies under a mean-variance framework is an open question.

We will seek an answer to this question, specifically of finding an efficient strategy for an AAI, with our insurer's surplus process assumed to follow a Brownian motion with drift, and investment being allowed in a standard Black-Scholes financial market with one risk-free asset and one risky asset, for simplicity. The level of ambiguity is weighted by a state-dependent preference parameter (to be introduced further below). We firstly formulate a robust control problem for the AAI under a mean-variance criterion. Secondly, we derive the closed-form expressions for an optimal benchmark strategy with reinsurance and investment, as well as the corresponding value function. Then, this result is used to solve the mean-variance problem for the AAI. Finally, some economic implications of our results and numerical illustrations are presented.

Comparing with the existing literature, we think that our paper has two main innovations:

(i) An optimal mean-variance problem for an AAI with reinsurance and investment is advanced the first time. Given how pervasive the understanding and use of mean-variance criteria is in the world of risky investments, this choice should be well-received by AAIs. We also derive explicit expressions for the efficient strategy, as well as for the corresponding value function. These provide easily implemented quantitative tools for the AAI's investment decision-making.

(ii) The impact of model-uncertainty robustness on mean-variance efficient strategies is investigated, which Zhang & Siu (2009) did not consider. Maenhout (2006) and Liu (2010) indicated that model uncertainty can be regarded as one aspect of risk source, and thus, when facing model uncertainty, investors should reduce their risk exposure undoubtedly. However, under a mean-variance criterion, we find that the AAI should not always keep a reduced risk exposure in the markets compared with the ambiguity-neutral insurer (ANI). On the contrary, the AAI should even increase the risk exposure in some circumstances, including those in which past investment decisions have been beneficial. Under these circumstances, this is the reverse of the decisions dictated under the frameworks of utility functions in Maenhout (2006) and Liu (2010), and can be interpreted as explaining some investment behavior which could otherwise be considered irrational.

The remainder of this paper is organized as follows. In Section 2, the economy and assumptions are described. In Section 3, an optimal mean-variance problem for an AAI is presented. Section 4 transforms the mean-variance problem into a benchmark problem, and obtains closed-form expressions for the optimal strategy and the corresponding value function. In section 5, the result for benchmark problem is used to solve the mean-variance problem for the AAI. Consequently, the efficient strategy for the AAI is derived in this section. Section 6 proposes economic implications, and analyzes our results with numerical examples. Section 7 provides our conclusions, and proposes some promising extensions of our work.

2 Economy and assumptions

In this paper, we assume that trading in the reinsurance and financial markets is continuous, without taxes or transaction costs. Let $(\boldsymbol{\Omega}, \mathcal{F}, P)$ be a complete probability space with filtration $\{\mathscr{F}_t\}_{t\in[0,T]}$ generated by two related standard one-dimension Brownian motions $\{Z_1(t)\}$, $\{Z_3(t)\}$, where $\mathrm{Cov}(Z_1(t), Z_3(t)) = \rho t$ and T is the time horizon. Any decision made at time t is based on $\mathscr{F}_t$ which can be interpreted as the information available until time t.

2.1 Surplus process

If both reinsurance and investment are absent, the insurer's surplus process can be assumed to satisfy a diffusion approximation (DA) model:

$$dR_0(t) = \mu_0 dt + b dZ_3(t), \tag{1}$$

where μ_0 represents the premium return rate of the insurer; $b > 0$ can be understood as the volatility of the insurer's surplus; $\{Z_3(t)\}$ is a one dimensional standard Brownian motion. The DA model approximates the classical Cramér-Lundberg model well as each claim is relatively small compared to the size of the surplus, and this approximation has been widely used in the literature, such as Grandell (1991), Browne (1995), Promislow & Young (2005), Gerber & Shiu (2006), Chen et al. (2010), Zeng & Li (2012) and so on.

We assume that the insurer can control her insurance risk by purchasing proportional reinsurance or acquiring new business (by acting as a reinsurer of other insurers, see i.e. Bäuerle (2005)). For each $t \in [0, T]$, the proportional reinsurance or new business level is denoted by the value of risk exposure $q(t) \in [0, +\infty)$. When $q(t) \in [0, 1]$, it corresponds to a proportional reinsurance cover; in this case, reinsurance premium will be paid by the cedent at the rate of $(1 - q(t))\eta$, where $\eta \geq \mu_0$ is the premium return rate of the reinsurer; meanwhile, the insurer pays $100q(t)\%$ while the reinsurer pays the rest $100(1 - q(t))\%$ for each claim occurring at time t. When $q(t) \in (1, +\infty)$, this is interpreted as acquiring new business. We define the process of risk exposure $\{q(t) : t \in [0, T]\}$ as the reinsurance strategy, and the DA dynamics for the surplus process associated with such a strategy $\{q(t) : t \in [0, T]\}$ is given by

$$dR(t) = [\lambda + \eta q(t)]dt + bq(t)dZ_3(t), \tag{2}$$

where $\lambda = \mu_0 - \eta$.

2.2 Financial Market

The financial market we consider consists of one risk-free asset (e.g., a bond) and one risky asset (e.g., a stock). The price process $S_0(t)$ of the risk-free asset follows the ordinary differential equation (ODE)

$$dS_0(t) = rS_0(t)dt, \tag{3}$$

where $S_0(0) = s_0 > 0$ and $r > 0$ is the risk-free interest rate. The price process $S_1(t)$ of the risky asset evolves according to Geometric Brownian Motion (GBM)

$$dS_1(t) = S_1(t)\left[(r + u)dt + \sigma dZ_1(t)\right], \tag{4}$$

with $S_1(0) = s_1 > 0$. Here $u > 0$ and $\sigma > 0$ are the risk-premium and volatility, $Z_1(t)$ and $Z_3(t)$ are two one-dimensional standard Brownian motions mentioned at the beginning of section 2. In addition, $Z_3(t)$ can be rewritten as

$$dZ_3(t) = \rho dZ_1(t) + \rho_0 dZ_2(t), \tag{5}$$

where $\rho_0 = \sqrt{1-\rho^2}$ and $Z_2(t)$ is a standard Brownian motion which is independent of $Z_1(t)$.

2.3 Wealth process

In addition to the insurer's ability to purchase proportional reinsurance or acquire new business, we also assume she is allowed to invest her surplus in the risky financial assets over $t \in [0, T]$. We consider the case of a single risky asset S_1 defined above, in order to keep the mathematics to a manageable level of complexity. In practice, this means that our framework can be interpreted as allowing investment into a single index fund or other type of aggregate risky fund.

Our insurer's trading strategy is therefore a pair of scalar stochastic processes $\pi = \{q(t), l(t)\}_{t\in[0,T]}$, where $q(t)$ represents the value of risk exposure, as described above, and $l(t)$ is the dollar amount invested in the risky asset S_1 at time t. The remainder $W^\pi(t) - l(t)$ is invested in the risk-free asset S_0 defined above, where $W^\pi(t)$ is the wealth process associated with strategy π. When the initial wealth is w_0, the wealth process $W^\pi(t)$ can be described by the following stochastic differential equation (SDE) system

$$\begin{cases} dW^\pi(t) = & [\lambda + \eta q(t) + ul(t) + rW^\pi(t)]dt + [\sigma l(t) + b\rho q(t)]dZ_1(t) \\ & + bq(t)\rho_0 dZ_2(t), \\ W^\pi(0) = & w_0. \end{cases} \tag{6}$$

3 Robust control problem under mean-variance criterion

In Bäuerle (2005), an insurer is assumed to be an ANI, i.e. one who does not worry about model uncertainty. The ANI aims to find a strategy such that the expected terminal wealth satisfies $\mathrm{E}^P[W^\pi(T)] = K$ where K is a predetermined objective, while minimizing the variance of the terminal wealth $\mathrm{Var}^P W^\pi(T) = \mathrm{E}^P[W^\pi(T) - K]^2$ over all strategies π in a specific set $\tilde{\Pi}$ of admissible strategies (see note following Definition 1 for an explanation of $\tilde{\Pi}$). Thus the optimization problem for the ANI can be stated as

$$(MV) \qquad \min_{\pi\in\tilde{\Pi}} \ \mathrm{E}^P[W^\pi(T) - K]^2, \qquad \text{subject to } \ \mathrm{E}^P[W^\pi(T)] = K, \tag{7}$$

where

$$K > w_0 e^{rT} + \frac{\lambda}{r}(e^{rT} - 1). \tag{8}$$

The expectation is computed under the probability measure P that reflects the ANI's certainty about the model. The reason for requiring the lower bound (8) on the objective K is that if K were no bigger than this level, the insurer could easily exceed the objective with a risk-free strategy (put all the money in the risk-free asset and get not involved to reinsurance business), which would then have zero variance; this would make the minimization problem ill-posed mathematically and of no economic interest.

To incorporate the model uncertainty into the mean-variance problem for an ambiguity-averse insurer (AAI), we assume that our insurer's knowledge with ambiguity is described under the probability measure P, namely the reference probability (or model), but that she is skeptical about this reference model, and hopes to consider some alternative models. She seeks a robust optimal strategy, and her thinking will follow the lines of Anderson et al. (1999). Loosely speaking, the AAI takes the model P as her reference model, but she recognizes that it is only an approximation of the true model and also takes account of alternative models. She is willing to consider all alternative models which can be represented as probability measures Q which are equivalent to (share the same sets of measure 0 as) the original P; in other words, she considers all Q in the set of probability measures $\mathcal{Q}$ defined by

$$\mathcal{Q} := \{Q | Q \sim P\}. \tag{9}$$

We assume the AAI attains robustness by considering a worst-case scenario $Q^* \in \mathcal{Q}$ and a strategy π^* to guard against facing the worst-case scenario. We will determined these worst-case options in the following way: For every fixed admissible strategy π, we propose a measure $Q^*(\pi) \in \mathcal{Q}$ which provides the biggest possible penalized variance (worst-case model when strategy π is fixed); Note that this defines $Q^*(\cdot)$ as a function from the set of admissible strategies into $\mathcal{Q}$. Then, we minimize the resulting worst-case penalized variance over all admissible strategies π. This gives us the minimized worst-case value function, attained at a specific optimal strategy π^*. Finally, we say that our worst-case model Q^* is the one corresponding to π^*, namely $Q^* = Q^*(\pi^*)$.

The details of how to acquire an adequate and analytically tractable formula for the measure-valued function $Q^*(\cdot)$ are in Section 4; here we mention that since $\mathcal{Q}$ is a very large (non-compact) set, finding a non-infinite maximizer $Q^*(\pi)$ for any π requires using a way to penalize those models in $\mathcal{Q}$ which seems likely the negative term in (10) below. In the meantime, assuming $Q^*(\cdot)$ has been determined, or more generally, for any $\mathcal{Q}$-valued map $Q^*(\cdot)$ defined on strategies, we may define the set of admissible strategies relative to it.

Definition 1 A trading strategy $\pi = \{q(t), l(t)\}_{t \in [0,T]}$ is said to be admissible, if

(i) $\forall t \in [0, T]$, $q(t) \geq 0$ and $Q^*(\pi) \in \mathcal{Q}$,

(ii) $\forall w_0 \in \mathbb{R}$, the corresponding SDE (6) has a pathwise unique solution $W^\pi(t)$,

(iii) the progressively measurable pair π satisfies $\mathrm{E}^{Q^*(\pi)}\left[\int_0^T \|\pi\|^4 dt\right] < \infty$ and $\mathrm{E}^{Q^*(\pi)}\left[\int_0^T |W^\pi(t)|^4 dt\right] < \infty$.

Denote by Π the set of all admissible strategies.

Note that this definition of admissibility requires that $Q^*(\pi)$ be absolutely continuous w.r.t. P. Also note that the set of admissible strategies $\tilde{\Pi}$ we mentioned on Page 6 can be taken as in the above definition replacing $Q^*(\pi)$ by fixed P therein.

Thus, we formulate a robust mean-variance problem inspired by Anderson et al. (2003) and Maenhout (2004) to modify problem (MV) as follow

$$(RMV) \qquad \min_{\pi\in\Pi} \; \mathrm{E}^{Q^*(\pi)}\left\{[W^\pi(T)-K]^2 - \int_0^T \frac{1}{\phi(s,w)} D(P, Q^*(\pi))ds\right\}, \tag{10}$$

$$\text{subject to} \qquad \mathrm{E}^{Q^*(\pi)}[W^\pi(T)] = K,$$

where $Q^*(\pi)$ will be defined later as a worst-case-scenario model which is dependent on π (see definition in (17)), $\mathrm{E}^{Q^*(\pi)}$ represents the expectation under $Q^*(\pi)$ for every admissible $\pi \in \Pi$, the penalization term $D(P, Q^*(\pi))$, also a function of π, is a measure of discrepancy between the probability measures P and $Q^*(\pi)$, [1] and the penalty factor ϕ is a function of time and wealth, i.e. of the pair (t, w), for which we provide a description on Page 10.

In the remainder of this Section, fixing a strategy $\pi \in \Pi$ for the insurer, we give some background material and explanations of how to define an element of $\mathcal{Q}$, and what the resulting dynamics of the wealth process become for the fixed π and a fixed $Q \in \mathcal{Q}$.

According to the celebrated (Cameron-Martin-)Girsanov theorem, the set of all $Q \in \mathcal{Q}$ can be represented as the set of models that differ from the 2-dimensional vector of correlated models (2) and (4) by the addition of a (bounded variation) drift term; more specifically, it turns out that those Q's are those for which there exists a progressively measurable pair of processes $\theta(t) = (\theta_1, \theta_2)$ such that

$$\frac{dQ}{dP} = \nu(T). \tag{11}$$

where $\nu(t) = \exp\left\{\int_0^t \theta_1 dZ_1(s) + \int_0^t \theta_2 dZ_2(s) - \frac{1}{2}\int_0^t (\theta_1^2 + \theta_2^2)ds\right\}$ and this process ν is a (Z_1, Z_2)-martingale under P. To ensure the martingale property, in this paper, we assume that $\theta(t)$ satisfies bounded condition

$$\exists\; constant\; C > 0, \forall t \in [0,T], \|\theta(t)\|^2 < C, a.s.. \tag{12}$$

[1] In fact, we will try to construct a map $Q^*(\pi)$ from π to $\mathcal{Q}$, see the rest of Section 3 and the beginning of Section 4

This then ensures that ν is a martingale under P relative to the filtration $\{\mathscr{F}_t\}_{t\in(0,T)}$ of the pair (Z_1, Z_2), and will allow us to perform computations under P and under all the Q in $\mathcal{Q}$ which satisfy (12).

It may seem that the boundedness of condition (12) is a restrictive technical assumption, e.g. one may wonder whether it was introduced for the sole purpose of applying Lemma 1[2], thereby making it easy to solve problem (RMV), at the cost of imposing an artificial constraint. We will see that this is not the case. Condition (12) is actually not a restriction in terms of the AAI's decision, since we will find that the worst-case scenario's measure Q^* features a bounded θ^*. This feature is specific to our setup, in which we use a Black-Scholes market, and the ambiguity-aversion preference parameter ϕ introduced by Maenhout (2004, 2006). A more complicated market, e.g. including a stochastic volatility, would not have this feature, and would require more technical difficulties to establish existence and uniqueness for problem (RMV).

We denote the set of θ satisfying (12) by Θ. Furthermore, as alluded to above, by Girsanov's theorem, the 2-dimensional Brownian motion differential $(dZ_1(t), dZ_2(t))$ to which one adds the drift term $-\theta(t)\,dt$, i.e. the pair of processes

$$dZ_1^Q(t) = dZ_1(t) - \theta_1(t)dt, \tag{13}$$
$$dZ_2^Q(t) = dZ_2(t) - \theta_2(t)dt, \tag{14}$$

has the law of standard 2-dimensional Brownian motion under the $Q \in \mathcal{Q}$ defined by its Radon-Nykodym derivative $\frac{dQ}{dP} = \nu(T)$ given above. Abusing notation slightly, we now use the letter $\mathcal{Q}$ to denote those measures Q as above in which $\theta \in \Theta$, i.e. θ satisfies condition (12). By using Girsanov's theorem to define this set of alternative models which the AAI may take into account, we are achieving a description of model uncertainty by allowing the drift parameters in the reinsurance market and the financial market to be undetermined. This is most useful in practice, because it is a notorious fact in financial markets, particularly in questions of portfolio selection, that returns, which are determined solely by drift parameters like θ above, are difficult to estimate with any reasonable precision.

In particular, for a given admissible π and $Q \in \mathcal{Q}$, inserting (13) and (14) into (6), the wealth process under the alternative model Q can be described as that which satisfies the following dynamics:

$$\begin{aligned} dW^\pi(t) = {} & [ul(t)+rW^\pi(t) + \lambda + q(t)\eta + \theta_1(t)(\sigma l(t) + b\rho q(t)) + b\rho_0\theta_2(t)q(t)]dt \\ & + [\sigma l(t) + b\rho q(t)]dZ_1^Q(t) + b\rho_0 q(t)dZ_2^Q. \end{aligned} \tag{15}$$

We notice that the wealth process under the alternative model in the class $\mathcal{Q}$ differs only in the drift term, as it should.

[2] Lemma 1 is based on bounded $\theta(t)$, and will be used to verify the viscosity solution technically, see Page 16 for details.

We may also be more specific about what the map $Q^*(\cdot)$ looks like. Since every $Q \in \mathcal{Q}$ is defined via a pair of drift processes (θ_1, θ_2) as in (13)-(14), our map $Q^*(\cdot)$ will be determined by a map taking any given strategy π to a pair of processes (θ_1, θ_2) satisfying (12).

We finish this section with comments on penalization. We follow Hansen & Sargent (2001), and measure the discrepancy between P and Q as the function

$$D(P, Q) := \frac{1}{2}(\theta_1^2 + \theta_2^2)$$

(see, e.g., Dupuis & Ellis (1997)). For problem (RMV), the AAI will choose one of alternative models as a worst-case scenario by maximizing the variance of the terminal wealth. Furthermore, she is well aware of the fact that the reference model is statistically the best representation of the existing data, thus large penalties are incurred for alternative models when they deviate far from the reference model. According to Maenhout (2004), $D(P, Q)$ measures the discrepancy between the reference model and an alternative model, and the function ϕ represents a preference parameter for ambiguity aversion, which measures the degree of confidence in the reference model. Stated in another way, the magnitude of the deviation penalty depends on the preference parameter.

In any case, extreme choices of aversion parameters easily lead to the following information. In the case $\phi = 0$, the insurer is entirely convinced that the true model is P, since any deviation from P will be penalized infinitely heavily by $\frac{1}{\phi}D(P, Q)$. Thus, $Q^*(\pi)$ should be chosen as P, which yields $D(P, Q^*(\pi)) = 0$ to guarantee $\frac{1}{\phi}D(P, Q^*(\pi)) = 0$. Problem (10) then degenerates to problem (7) where the AAI becomes an ANI. At the opposite end of the spectrum, in the case $\phi = \infty$, the insurer has no information about the true model. Thus, the penalty vanishes, which implies that the insurer should consider all alternative models on equal footing, ignoring any degree of confidence (see Uppal & Wang (2003)).

4 Optimal strategy under benchmark criterion

Since problem (RMV) is an optimization problem with a constraint, a Lagrange multiplier $\mathcal{L} \in \mathbb{R}$ can be introduced to tackle the equality constrain $\mathrm{E}^{Q^*(\pi)}[W^\pi(T)] = K$. By using this approach, for any given $Q^*(\pi)$, problem (RMV) can be solved via following Lagrangian dual problem

$$\begin{aligned}(DP) \quad \max_{\mathcal{L} \in R} \min_{\pi \in \Pi} \quad \mathrm{E}^{Q^*(\pi)}\Big\{ [W^\pi(T) - K]^2 - \int_0^T \frac{1}{\phi} D(P, Q^*(\pi)) ds \\ +2\mathcal{L}[W^\pi(T) - K] \Big\}.\end{aligned} \tag{16}$$

To solve problem (DP), the interior minimization problem in (DP) is immediately seen to be equivalent to

$$\min_{\pi\in\Pi} \ \mathrm{E}^{Q^*(\pi)}\left\{[W^\pi(T)-K+\mathcal{L}]^2-\int_0^T\frac{1}{\phi}D(P,Q^*(\pi))ds-\mathcal{L}^2\right\}.$$

Under fixed $\mathcal{L}$, we may first consider the following benchmark problem

$$(RBM)\qquad \min_{\pi\in\Pi} \ \mathrm{E}^{Q^*(\pi)}\left\{[W^\pi(T)-B]^2-\int_0^T\frac{1}{\phi}D(P,Q^*(\pi))ds\right\},$$

where $B:=K-\mathcal{L}$ can be regarded as a benchmark for each fixed $\mathcal{L}$.

We are now ready to define the alternative model which we take to describe a worst-case scenario:

$$Q^*(\pi):=\arg\max_{Q\in\mathcal{Q}} \ \mathrm{E}^{Q}\left\{[W^\pi(T)-B]^2-\int_0^T\frac{1}{\phi}D(P,Q)ds\right\}. \tag{17}$$

As mentioned in Section 3, this means that for problem (RMV), by defining our worst-case model as $Q^*:=Q^*(\pi^*)$, we are first finding a worst-case scenario probability measure function $Q^*(\pi)$ for each fixed π under problem (RBM), and then optimizing over π to get the π^* which attains the minimal penalized variance in for problem (RBM). This is a legitimate way to proceed in terms of the original problem (RMV), as long as the minimum is attained, and even more crucially, as long as the final measure $Q^*(\pi^*)$ that we obtain via (RBM) does not depend on the Lagrange multiplier $\mathcal{L}$. Fortunately, with our particular setup, we are able to verify that our $Q^*(\pi^*)$ has this feature (see Remark 2).

Under the above definition of $Q^*(\pi)$, problem (RBM) can be rewritten as

$$\min_{\pi\in\Pi}\max_{Q\in\mathcal{Q}}\mathrm{E}^{Q}\left[[W^\pi(T)-B]^2-\int_0^T\frac{1}{\phi}D(P,Q)ds\right]. \tag{18}$$

This means that we may adopt a stochastic dynamical programming approach to solve problem (18). Define the corresponding value function for problem (18) as

$$J(t,w)=\min_{\pi\in\Pi}\max_{Q\in\mathcal{Q}}\mathrm{E}^{Q}_{t,w}\left[[W^\pi(T)-B]^2-\int_t^T\frac{1}{\phi}D(P,Q)ds\right], \tag{19}$$

where $\mathrm{E}^Q_{t,w}[\cdot]=\mathrm{E}^Q[\cdot\mid W(t)=w]$. To solve this problem, we follow Theorem 3.4 in Talay & Zheng (2002) (see Lemma 1 on Page 16) to establish and solve the corresponding HJB equation, also known in this case as a so-called Hamilton-Jacobi-Bellman-Isaacs (HJBI)) equation, which is

$$\min_{\pi\in\Pi}\max_{\theta_1,\theta_2\in\Theta}\left\{\mathcal{A}^\pi J-\frac{1}{2\phi}(\theta_1^2+\theta_2^2)\right\}=0, \tag{20}$$

with the boundary condition $J(T,w) = (w-B)^2$, where differential operator $\mathcal{A}^\pi$ is computed as

$$\begin{aligned}\mathcal{A}^\pi J = J_t + J_w[ul(t)+wr + \lambda + \eta q(t) + \theta_1\sigma l(t) + \theta_1 b\rho q(t) + b\rho_0\theta_2 q(t)]\\ + \frac{1}{2}J_{ww}\left[(\sigma l(t) + b\rho q(t))^2 + b^2\rho_0^2 q^2(t)\right].\end{aligned} \tag{21}$$

Here, J_t, J_w, J_{ww} represent the partial derivatives of the value function w.r.t. the corresponding variables. Following the idea suggested by Maenhout (2004, 2006), we have to impose a "homothetic" preference parameter $\phi > 0$, which renders problem (RBM) analytically tractable, and ensures that the penalty in problem (RBM) is reasonable. Maenhout (2004) indicates the preference parameter should change with the state variable (in our case, the state variable is the current wealth w). For example, it is stands to reason that the AAI would have more robustness when her economic condition is favorable. Specifically, we follow Maenhout (2004, 2006) to choose the preference parameter ϕ as

$$\phi(t,w) = \frac{\beta}{J(t,w)} > 0. \tag{22}$$

This can be interpreted in the following way: Lower values of J correspond to a favorable condition under the benchmark criterion, and this leads to a greater preference parameter ϕ, which implies the AAI would consider more robustness in model uncertainty, i.e. would be more willing to take on higher levels of model uncertainty risk, thereby minimizing her reliance on parametric models. Although we do not know the precise form of J until we solve the problem, it is not hard to conjecture that the structure of J is a polynomial in w with undetermined functions of time as its coefficients. See Page 959-961 in Maenhout (2004) for further detail, also see Maenhout (2006), Liu (2010), Branger et al. (2013) for "Homothetic" robustness.

Remark 1 The preference parameter $\phi = \beta/J$ can be interpreted as the individual preference for ambiguity-aversion, where $\beta > 0$ is the ambiguity-aversion level describing individual attitude to model uncertainty.

In order to solve HJB equation (20), we first propose an *ansatz* for the structure of the value function, in which variables are separated. Then, we aim to derive the drift for the worst-case scenario $Q^*(\pi)$ and the optimal strategy π^* under $Q^*(\cdot)$. Finally, we hope that the variables can be separated and solved explicitly to verify the ansatz.

Step 1: propose the ansatz.

We conjecture that the value function has the following structure

$$J(t,w) = L(t)w^2 + M(t)w + N(t), \tag{23}$$

where $L(t)$, $M(t)$ and $N(t)$ are three functions to be determined. The assumption $J_{ww} > 0$ implies $L(t) > 0$ for all $t \in [0,T]$, and boundary condition $J(T,w) = (w-B)^2$ implies that

$$L(T) = 1, \quad M(T) = -2B, \quad N(T) = B^2. \tag{24}$$

A direct calculation yields

$$J_t = L_t w^2 + M_t w + N_t, \quad J_w = 2Lw + M, \quad J_{ww} = 2L. \tag{25}$$

Step 2: derive the worst-case drifts and optimal strategy.

Differentiating (20) w.r.t. θ_1 and θ_2 to maximize over Q, the first-order conditions are

$$\theta_1^*(t,w) = J_w[\sigma l(t) + b\rho q(t)]\phi(t,W), \quad \theta_2^*(t,w) = J_w b\rho_0 q(t)\phi(t,w). \tag{26}$$

Substituting (22) and (25) into (26), we have

$$\theta_1^*(t,w) = \frac{2\beta L(t)[\sigma l(t) + b\rho q(t)]}{2wL(t) + M(t)}, \quad \theta_2^*(t,w) = \frac{2b\rho_0\beta L(t)q(t)}{2wL(t) + M(t)}. \tag{27}$$

The drift terms θ_1^* and θ_2^* govern the worst-case scenario which is considered by the AAI, and the optimal strategy will be derived under this alternative model to attain the robustness. Inserting (27) into HJB equation (20) yields

$$\begin{aligned}\min_{\pi\in\Pi} \big\{ & L_t w^2 + M_t w + N_t + (2Lw + M)[ul(t) + wr + \lambda + \eta q(t)] \\ & + (\beta+1)L[\sigma^2 l^2(t) + 2\sigma b\rho q(t)l(t) + b^2 q^2(t)]\big\} = 0\end{aligned} \tag{28}$$

According to the first-order conditions[3] for $\pi = \{q(t), l(t)\}_{t\in[0,T]}$, we have

$$q^*(t,w) = \frac{(ub\rho - \eta\sigma)}{b^2\sigma\rho_0^2(1+\beta)}\left(w + \frac{M(t)}{2L(t)}\right), \quad l^*(t,w) = \frac{(\sigma\rho\eta - bu)}{b\sigma^2\rho_0^2(1+\beta)}\left(w + \frac{M(t)}{2L(t)}\right). \tag{29}$$

To consider the constraint $q \geq 0$, we need to separate the plane (t,w) into the following two regions:

$$\mathcal{A}_1 = \left\{(t,w) \in \bar{\mathcal{O}}, q^*(t,w) \geq 0\right\},$$
$$\mathcal{A}_2 = \left\{(t,w) \in \bar{\mathcal{O}}, q^*(t,w) < 0\right\},$$

where $\mathcal{O} := (0,T) \times \mathbb{R}$ and $\bar{\mathcal{O}}$ denotes the closure of $\mathcal{O}$.

Step 3: separate and derive the variables.

Due to the form of $q^*(t,w)$, we discuss the following two cases, respectively.

(1) The case of $ub\rho - \eta\sigma \geq 0$.

(i) If $(t,w) \in \mathcal{A}_1$, the candidate optimal strategy (q^*, l^*) is allowed. Inserting (29) into (28), we obtain

$$LL_t w^2 + wLM_t + LN_t + (2wL^2 + LM)(wr + \lambda) - A(2wL + M)^2 = 0, \tag{30}$$

where

$$A = \frac{(bu - \rho\sigma\eta)^2 + \rho_0^2\sigma^2\eta^2}{b^2\sigma^2\rho_0^2(\beta+1)} > 0. \tag{31}$$

---

[3] According to the assumption $J_{ww} > 0$, one can easily verify that the second-order conditions can be fulfilled to ensure the minimization.

By separating the terms with w^2, with w and without w, in order to ensure (30), we only need that the following system of ODEs is satisfied:

$$\begin{cases} LL_t + 2rL^2 - AL^2 = 0, \quad L(T) = 1, \\ LM_t + rLM + 2\lambda L^2 - AL^2 = 0, \quad M(T) = -2B, \\ LN_t + \lambda LM - \dfrac{AM^2}{4}, \quad N(T) = B^2, \end{cases} \tag{32}$$

Direct calculation yields the solution to ODEs (32) as

$$L(t) = e^{-(A-2r)(T-t)}, \quad M(t) = -2g(t)e^{-(A-r)(T-t)}, \quad N(t) = g^2(t)e^{-A(T-t)}, \tag{33}$$

where

$$g(t) = B + \frac{\lambda}{r}\left[1 - e^{r(T-t)}\right]. \tag{34}$$

Thus the regions $\mathcal{A}_1$ and $\mathcal{A}_2$ can be rewritten as

$$\mathcal{A}_1 = \left\{(t,w) \in [0,T] \times \mathbb{R}, w - g(t)e^{-r(T-t)} > 0\right\},$$
$$\mathcal{A}_2 = \left\{(t,w) \in [0,T] \times \mathbb{R}, w - g(t)e^{-r(T-t)} \leq 0\right\}.$$

(ii) If $(t,w) \in \mathcal{A}_2$, the candidate for optimal strategy (29) is not allowed due to the fact that $q < 0$ is not allowed. Notice that the left side of (28) is a decreasing function w.r.t. q in the interval $[0, +\infty)$; this implies that we obtain $q^*(t,w) = 0$ and insert it into (29). By the first-order condition for $l(t,w)$, we arrive at

$$q_2^*(t,w) = 0, \quad l_2^*(t,w) = -\frac{u}{\sigma^2(1+\beta)}\left(w + \frac{\tilde{M}(t)}{2\tilde{L}(t)}\right), \tag{35}$$

and the corresponding value function is $J(t,w) = \tilde{L}(t)w^2 + \tilde{M}(t)w + \tilde{N}(t)$, $\forall(t,w) \in \mathcal{A}_2$.

Inserting (35) into (28) and separating the terms with w^2, w and without w, we arrive at the following ODEs:

$$\begin{cases} \tilde{L}\tilde{L}_t + 2r\tilde{L}^2 - A_1\tilde{L}^2 = 0, \quad \tilde{L}(T) = 1, \\ \tilde{L}\tilde{M}_t + r\tilde{L}\tilde{M} + 2\lambda\tilde{L}^2 - A_1\tilde{L}^2 = 0, \quad \tilde{M}(T) = -2B, \\ \tilde{L}\tilde{N}_t + \lambda\tilde{L}\tilde{M} - \dfrac{A_1\tilde{M}^2}{4}, \quad \tilde{N}(T) = B^2, \end{cases} \tag{36}$$

where

$$A_1 = \frac{u^2}{\sigma^2(\beta+1)} > 0. \tag{37}$$

Direct calculation yields

$$\tilde{L}(t) = e^{-(A_1-2r)(T-t)}, \quad \tilde{M}(t) = -2g(t)e^{-(A_1-r)(T-t)}, \quad \tilde{N}(t) = g^2(t)e^{-A_1(T-t)}, \tag{38}$$

where $g(t)$ is given by (34). Moreover, we let $C^{1,2}(\bar{\mathcal{O}})$ be the set of all functions $\varphi : [0,T]\times\mathbb{R} \to \mathbb{R}$ such that φ_t, φ_w and φ_{ww} are all continuous in (t, w). Denote the boundary between $\mathcal{A}_1$ and $\mathcal{A}_2$ as

$$\mathcal{A}_0 = \left\{(t, w) \in [0, T] \times \mathbb{R}, w - g(t)e^{-r(T-t)} = 0\right\}.$$

This generates a difficulty for $J(t, w)$: It cannot be considered as a classical solution to the HJB equation (20); indeed, since $L(t) \neq \tilde{L}(t)$ on $\mathcal{A}_0$, we cannot say that $J(t, w) \in C^{1,2}(\bar{\mathcal{O}})$. Therefore, because of the discontinuity of $J(t, w)$ in the parameter t on the set $\mathcal{A}_0$, we may and must assert that $J(t, w)$ satisfies (20) in the sense of viscosity solution. To be specific, we recall what this means.

Definition 2 $J \in C(\bar{\mathcal{O}})$ is called a viscosity subsolution to (20) at the fixed value $(t, w) \in \bar{\mathcal{O}}$, if for any test function $\varphi \in C^{1,2}(\bar{\mathcal{O}})$ such that $J(t, w) = \varphi(t, w)$, whenever $J - \varphi$ attains a local maximum at $(t, w) \in \mathcal{O}$, we have

$$\min_{\pi\in\Pi}\max_{\theta_1,\theta_2\in\Theta}\left\{\mathcal{A}^\pi\varphi - \frac{\varphi_w^2}{2\beta\varphi_{ww}}(\theta_1^2+\theta_2^2)\right\} \geq 0; \tag{39}$$

similarly, $J \in C(\mathcal{O})$ is called a viscosity supersolution to (20) if the inequality "$\geq$" is changed to "$\leq$" in (39) and "local maximum" is changed to "local minimum". Finally, $J(t, w) \in C(\mathcal{O})$ is a viscosity solution to (20), if it is both a viscosity subsolution and viscosity supersolution.

Remark 2 Inserting the optimal strategy (29) into the worst-case drift (27), we notice that the optimal drifts are independent of $\mathcal{L}$, whatever pair (t, w) is in $\mathcal{A}_1$ or $\mathcal{A}_2$. This feature reveals that Q^* is independent of the benchmark B: It only depends on the parameters of reinsurance market and financial market.

(2) The case of $ub\rho - \eta\sigma < 0$. Using the same approach as the case of $ub\rho - \eta\sigma \geq 0$, we can obtain the similar results.

Combining the situations of two cases and Definition 2, the viscosity solution to the HJB equation (20) can be described as following theorem.

Theorem 1 *In the case $ub\rho - \eta\sigma \geq 0$, a viscosity solution to HJB equation (20) is given by*

$$J(t, w) = \begin{cases} L(t)w^2 + M(t)w + N(t), & \text{if } (t, w) \in \mathcal{A}_1, \\ \tilde{L}(t)w^2 + \tilde{M}(t)w + \tilde{N}(t) & \text{if } (t, w) \in \mathcal{A}_2. \end{cases}$$

In the case $ub\rho - \eta\sigma < 0$, a viscosity solution to HJB equation (20) is given by

$$J(t, w) = \begin{cases} L(t)w^2 + M(t)w + N(t), & \text{if } (t, w) \in \mathcal{A}_2, \\ \tilde{L}(t)w^2 + \tilde{M}(t)w + \tilde{N}(t) & \text{if } (t, w) \in \mathcal{A}_1, \end{cases}$$

where $L(t)$, $M(t)$, $N(t)$ and $\tilde{L}(t)$, $\tilde{M}(t)$, $\tilde{N}(t)$ are given in (33) and (38), respectively.

Proof We only prove the case $ub\rho-\eta\sigma \geq 0$, the case $ub\rho-\eta\sigma < 0$ can be proved similarly. As mentioned above, $\mathcal{A}_0$ is the only region where the nonsmoothness occurs. Thus, we have to find a solution in the sense of viscosity solutions. According to Definition 2, let $\varphi \in C^{1,2}(\bar{\mathcal{O}})$ be such that $J-\varphi$ attains a local maximum at $(t,w) \in \mathcal{A}_0$. We arrive at $\varphi = \varphi_t = \varphi_w = 0$ and $\varphi_{ww} \geq 2L(t)$ to verify the following inequalities

$$\begin{aligned}
&\min_{\pi\in\Pi}\max_{\theta_1,\theta_2}\left\{\mathcal{A}^\pi\varphi - \frac{\varphi_w^2}{2\beta\varphi_{ww}}(\theta_1^2+\theta_2^2)\right\}\\
&\quad= \min_{\pi\in\Pi}\max_{\theta_1,\theta_2}\left\{\frac{1}{2}\varphi_{ww}\left[(\pi\sigma+b\rho q(t))^2+b^2\rho_0^2q^2(t)\right]\right\}\\
&\quad= \min_{\pi\in\Pi}\left\{\frac{1}{2}\varphi_{ww}\left[(\pi\sigma+b\rho q(t))^2+b^2\rho_0^2q^2(t)\right]\right\}\\
&\quad\geq \min_{\pi\in\Pi}\left\{L(t)\left[(\pi\sigma+b\rho q(t))^2+b^2\rho_0^2q^2(t)\right]\right\} \geq 0.
\end{aligned}$$

Therefore, $J(t,w)$ is a viscosity subsolution to the HJB equation (20). We can prove that it is also a viscosity supersolution by definition, and hence a viscosity solution to the HJB equation (20). □

For problem (RBM), one could apply Theorem 3.4 in Talay & Zheng (2002) to verify the viscosity solution to HJB equation (20) is indeed the value function of problem (RBM), which can be shown as

Lemma 1 *Assume that F is a continuous function such that*

$$|F(w)-F(\overline{w})| \leq V(|w|,|\overline{w}|)(|w-\overline{w}|),$$

where $V(|w|,|\overline{w}|)$ is a polynomial function. Then the value function $J(t,w)$ defined in (19) *is the unique viscosity solution in the space*

$$\begin{aligned}
S := \Bigg\{&\psi(t,w) \text{ is continuous on } [0,T]\times\mathbb{R}; \exists\overline{A}>0,\\
&\lim_{w^2\to\infty}\psi(t,w)\exp(-\overline{A}(\log w)^2)=0, \forall t\in[0,T]\Bigg\}
\end{aligned}$$

to the HJB equation (20) with boundary condition $J(T,w)=F(w)$.

To establish Lemma 1 one only needs to follow the method in the proof of Theorem 3.4 in Talay & Zheng (2002). The only difference between our problem and the problem in Talay & Zheng is that in our case the value function defined in (19) has an extra term $\int_t^T \frac{J}{\beta}\left(\theta_1^2+\theta_2^2\right)ds$. Notice that, thanks to Theorem 1, J is a deterministic quadratic function of w, and θ is bounded. Consequently our extra term does not create any difficulties when applying the proof method of Talay & Zheng (2002). One easily verifies that $F(w)=(w-B)^2$ satisfies the condition in Lemma 1 and that the viscosity solution J given in Theorem 1 belongs to the space S. Further details of how to establish Lemma 1 are omitted. Lemma 1 now implies that J in Theorem

1 is the unique viscosity solution to (20) with boundary condition $J(T, w) = (w - B)^2$, which implies that it is the value function we defined by (19). We thus arrive at the following theorem.

Theorem 2 *Problem (RBM) with preference parameter $\phi(t, w) = \frac{\beta}{J(t,w)}$, where $J(t, w)$ is shown in Theorem 1, for the case $ub\rho - \eta\sigma \geq 0$, has an optimal strategy given by*

$$\pi^*(t, w) = \begin{cases} \left(\frac{(ub\rho - \eta\sigma)}{b^2\sigma\rho_0^2(1+\beta)} \left(w - g(t)e^{-r(T-t)}\right), \frac{(\sigma\rho\eta - bu)}{b\sigma^2\rho_0^2(1+\beta)} \left(w - g(t)e^{-r(T-t)}\right) \right) & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (t, w) \in \mathcal{A}_1, \\ \left(0, -\frac{u}{\sigma^2(1+\beta)} \left(w - g(t)e^{-r(T-t)}\right)\right), \qquad\qquad \text{if } (t, w) \in \mathcal{A}_2. \end{cases}$$

For the case $ub\rho - \eta\sigma < 0$, an optimal strategy is given by

$$\pi^*(t, w) = \begin{cases} \left(\frac{(ub\rho - \eta\sigma)}{b^2\sigma\rho_0^2(1+\beta)} \left(w - g(t)e^{-r(T-t)}\right), \frac{(\sigma\rho\eta - bu)}{b\sigma^2\rho_0^2(1+\beta)} \left(w - g(t)e^{-r(T-t)}\right) \right), & \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{if } (t, w) \in \mathcal{A}_2, \\ \left(0, -\frac{u}{\sigma^2(1+\beta)} \left(w - g(t)e^{-r(T-t)}\right)\right), \qquad\qquad \text{if } (t, w) \in \mathcal{A}_1, \end{cases}$$

where $g(t)$ is given by (34).

Remark 3 We define a stopping time $\tau = \inf\left\{s \geq 0 : W^*(s) - g(s)e^{r(s-T)} = 0\right\}$ with $W^*(t) := W^{\pi^*}(t)$ for simplicity. If $\tau \leq T$, the optimal strategy for the AAI at time τ is $(0, 0)$. If $T \geq t > \tau$, according to the dynamic wealth process (6), $W^*(t)$ would vary along the trajectory $W^*(t) - g(t)e^{r(t-T)} = 0$. This circumstance would imply that the insurer may keep reserving all the money in the risk-free asset and spread all insurance risks to the reinsurer over $[\tau, T]$. At time T, she will obtain a deterministic terminal wealth B.

Said differently, the AAI will change her optimal strategy to $(0, 0)$ if the pair $(t, W^*(t))$ hits region $\mathcal{A}_0$. For example, if $ub\rho - \eta\sigma \geq 0$ and $(0, w_0) \in \mathcal{A}_1$, the AAI will maintain the optimal strategy (29) until $(t, W^*(t))$ hits $\mathcal{A}_0$. The optimal strategy in this situation can be rewritten as follow

$$\pi^*(t) = \begin{cases} \left(\frac{(ub\rho - \eta\sigma)}{b^2\sigma\rho_0^2(1+\beta)} \left(W^*(t) - g(t)e^{-r(T-t)}\right), \frac{(\sigma\rho\eta - bu)}{b\sigma^2\rho_0^2(1+\beta)} \left(W^*(t) - g(t)e^{-r(T-t)}\right) \right), & \\ \qquad 0 \leq t < \tau \wedge T, \\ (0, 0), \quad \tau \wedge T \leq t \leq T. \end{cases}$$

5 Efficient strategy under mean-variance criterion

In this section, we investigate problem (RMV) for the AAI based on the previous results. Putting $(0, w_0)$ into the value function $J(t, w)$, a duality theorem (see Bai & Zhang (2008)) connects problem (RMV) to problem (RBM) via

$$J_{RMV}(0, w_0) = J(0, w_0 : \mathcal{L}^*) = \max_{\mathcal{L} \in \mathbb{R}} J(0, w_0 : \mathcal{L}), \tag{40}$$

where $J_{RMV}(t,w)$ is the corresponding value function for problem (RMV). We only analyze the case $ub\rho-\eta\sigma\geq 0$:

$$J(0,w_0:\mathcal{L})=\begin{cases} e^{-AT}\left[e^{rT}w_0+\frac{\lambda}{r}(e^{rT}-1)-K+\mathcal{L}\right]^2-\mathcal{L}^2, \\ \quad \text{if } \mathcal{L}\geq K-e^{rT}w_0-\frac{\lambda}{r}(e^{rT}-1), \\ e^{-A_1T}\left[e^{rT}w_0+\frac{\lambda}{r}(e^{rT}-1)-K+\mathcal{L}\right]^2-\mathcal{L}^2, \\ \quad \text{if } \mathcal{L}< K-e^{rT}w_0-\frac{\lambda}{r}(e^{rT}-1). \end{cases}$$

If $(0,w_0)\in\mathcal{A}_2$, the first-order condition for $\mathcal{L}$ implies it attains the maximum of J at

$$\mathcal{L}^*=\frac{\left[d-e^{rT}w_0-\frac{\lambda}{r}(e^{rT}-1)\right]e^{-A_1T}}{e^{-A_1T}-1}. \tag{41}$$

If $(0,w_0)\in\mathcal{A}_1$, $\mathcal{L}$ attains the maximum at $\mathcal{L}^*=K-e^{rT}w_0-\frac{\lambda}{r}(e^{rT}-1)$ resulting in an optimal strategy $(0,0)$, which fails to reach the given terminal wealth expectation K. Therefore, in the case $ub\rho-\eta\sigma\geq 0$, $\mathcal{L}$ can not attain a maximum if $(0,w_0)\in\mathcal{A}_1$.

Denote the variance with penalty PVar^Q for terminal wealth $W^\pi(T)$ as

$$\mathrm{PVar}^Q W^\pi(T)=\mathrm{E}^Q\left\{[W^\pi(T)-K]^2-\int_0^T\frac{1}{\phi}D(P,Q)ds\right\}.$$

Then we define the efficient strategy and the efficient frontier for problem (RMV) as follows.

Definition 3 An admissible strategy π^* with $\mathrm{E}^{Q^*}[W^*(T)]=K$ is called an efficient strategy for problem (RMV) if there exists no admissible strategy π such that $\mathrm{E}^{Q^*}[W^\pi(T)]=K$ and $\mathrm{PVar}^{Q^*}W^\pi(T)<\mathrm{PVar}^{Q^*}W^*(T)$ under chosen alternative model Q^*, where $\mathrm{PVar}^{Q^*}W^*(T)$ corresponds to the efficient variance with penalty. Moreover, $\left(\mathrm{PVar}^{Q^*}W^*(T),K\right)$ is called an efficient point and the set of all efficient points is called the efficient frontier.

Based on results of (RBM) and the duality theorem, the efficient strategy and efficient frontier can be obtained as the following theorem.

Theorem 3 *(1) In the case of $ub\rho-\eta\sigma\geq 0$ with $(0,w_0)\in\mathcal{A}_3$, the efficient frontier for problem (RMV) satisfies*

$$\mathrm{PVar}^{Q^*}W^*(T)=\frac{\left(\mathrm{E}^{Q^*}\left[W^*(T)\right]-e^{rT}w_0-\frac{\lambda}{r}(e^{rT}-1)\right)^2}{e^{A_1T}-1}, \tag{42}$$

where an efficient strategy for problem (RMV) can be presented as

$$\pi^*(t)=\begin{cases}\left(0,-\frac{u}{\sigma^2(1+\beta)}\left(W^*(t)-G_1(t)e^{-r(T-t)}\right)\right), & 0\leq t\leq\tau_1\wedge T,\\ (0,0), & \tau_1\wedge T<t\leq T,\end{cases}$$

and

$$\mathcal{A}_3 = \left\{(t,w) \in [0,T] \times \mathbb{R}, w - G_1(t)e^{-r(T-t)} \leq 0\right\},$$
$$\mathcal{A}_4 = \left\{(t,w) \in [0,T] \times \mathbb{R}, w - G_1(t)e^{-r(T-t)} > 0\right\},$$

with $G_1(t) = K - \frac{\left[K - e^{rT}w_0 - \frac{\lambda}{r}(e^{rT}-1)\right]e^{-A_1 T}}{e^{-A_1 T}-1} + \frac{\lambda}{r}\left[1 - e^{r(T-t)}\right]$. *In addition,* A_1 *is given by (37) and* τ_1 *is given by*

$$\tau_1 = \inf\left\{s \geq 0 : W^*(s) - G_1(s)e^{-r(T-s)}\right\} = 0.$$

(2) In the case of $ub\rho - \eta\sigma < 0$ *with* $(0, w_0) \in \mathcal{A}_5$*, the efficient frontier satisfies*

$$\mathrm{PVar}^{Q^*} W^*(T) = \frac{\left(\mathrm{E}^{Q^*}[W^*(T)] - e^{rT}w_0 - \frac{\lambda}{r}(e^{rT}-1)\right)^2}{e^{AT}-1}, \tag{43}$$

and an efficient strategy can be presented as

$$\pi^*(t,w) = \begin{cases} \left(\frac{(ub\rho-\eta\sigma)}{b^2\sigma\rho_0^2(1+\beta)}\left(w - G_2(t)e^{-r(T-t)}\right), \frac{(\sigma\rho\eta-bu)}{b\sigma^2\rho_0^2(1+\beta)}\left(w - G_2(t)e^{-r(T-t)}\right)\right), \\ \qquad\qquad\qquad 0 \leq t \leq \tau_2 \wedge T, \\ (0,0), \qquad\qquad \tau_2 \wedge T < t \leq T. \end{cases}$$

where

$$\mathcal{A}_5 = \left\{(t,w) \in [0,T] \times \mathbb{R}, w - G_2(t)e^{-r(T-t)} \leq 0\right\},$$
$$\mathcal{A}_6 = \left\{(t,w) \in [0,T] \times \mathbb{R}, w - G_2(t)e^{-r(T-t)} > 0\right\},$$

with $G_2(t) = K - \frac{\left[K - e^{rT}w_0 - \frac{\lambda}{r}(e^{rT}-1)\right]e^{-AT}}{e^{-AT}-1} + \frac{\lambda}{r}\left[1 - e^{r(T-t)}\right]$. *In addition,* A *is given by (31) and* τ_2 *is given by*

$$\tau_2 = \inf\left\{s \geq 0 : W^*(s) - G_2(s)e^{-r(T-s)}\right\} = 0.$$

(3) In the case of $ub\rho - \eta\sigma \geq 0$ *with* $(0, w_0) \in \mathcal{A}_4$ *and the case of* $ub\rho - \eta\sigma < 0$ *with* $(0, w_0) \in \mathcal{A}_6$*, there exists no efficient strategy for problem* (RMV).

6 Robustness for model uncertainty

This section is devoted to investigating the impact of model-uncertainty robustness on the AAI's decision. In the numerical illustrations, we set the basic markets parameters given by Table 1 unless otherwise stated. These parameters result in the case $ub\rho - \eta\sigma = -0.1996 < 0$, which implies the efficient strategy involves both reinsurance market and financial market at the beginning. We omit the economic analysis for the case $ub\rho - \eta\sigma \geq 0$, since it does not associate with any reinsurance over the investment-insurance horizon, and the conclusions are similar with the investment part in the case $ub\rho - \eta\sigma < 0$.

Table 1 Values of parameters

μ_0	η	b	r	u	σ	ρ	K	w_0	$T(years)$
0.7	0.8	1	0.05	0.02	0.25	0.02	2	1	5

6.1 Detection-error probabilities

We aim to find the quantitative effect of model uncertainty via numerical examples. Anderson et al. (2003) argued that β should be chosen in such a way that the worst-case scenario with β is difficult to distinguish from the reference model for the AAI. Specifically, as suggested by Anderson et al. (2003), we use a detection-error probability to verify that for a chosen ambiguity-aversion level β, it is reasonable to operate under a worst-case scenario. The heuristic is as follows. A parameter β with high detection-error probability indicates that the AAI is likely to select an incorrect model, when faced with the choice between using the worst-case-scenario model and using the reference model. In such a situation, it is safer to acknowledge one's ambiguity aversion, and go with the worsk-case scenario. Therefore, to justify using our model robustness framework, our detection-error probability should preferably be high. We now compute this probability.

According to Anderson et al. (2003) and Maenhout (2006), we can calculate the detection-error probability for our parameter setup as follows. Denote $\xi_{1,t}$ the log of Radon-Nikodym derivative $\frac{dQ^*}{dP}$ as

$$\xi_{1,t} = \log\left[\frac{dQ^*}{dP}\right] = \int_0^t \theta_1^* dZ_1(s) + \theta_2^* dZ_2(s) - \frac{1}{2}\int_0^t ({\theta_1^*}^2 + {\theta_2^*}^2)ds;$$

the detection-error probability of incorrectly selecting P over Q^*, or vice-versa, assuming a uniform prior selection of P or Q^*, is defined as

$$\xi_T(\beta) = \frac{1}{2}\text{Prob}(\xi_{1,T} > 0|P, \mathcal{F}_0) + \frac{1}{2}\text{Prob}(\xi_{1,T} < 0|Q^*, \mathcal{F}_0). \tag{44}$$

This formula stems from the fact that since $\frac{dQ^*}{dP}$ is a martingale with expectation 1 under P, when it is larger than 1, this indicates that Q^* is more likely than P, given that P was selected; a similar argument holds when Q^* is selected. A calculation yields θ_1^* and θ_2^* under Q^* as

$$\theta_1^* = \frac{\beta}{1+\beta}\cdot\frac{u}{\sigma}, \quad \theta_2^* = \frac{\beta}{1+\beta}\cdot\frac{ub\rho - \eta\sigma}{b\sigma\rho_0}. \tag{45}$$

Here we only consider $(0, w_0)$ in the region $\mathcal{A}_5$ since region $\mathcal{A}_6$ leads to zero-drifts, which does not enter into the consideration of model uncertainty. The constants θ_1^* and θ_2^* lead to an explicit expression for $\xi_T(\beta)$ as

$$\xi_T(\beta) = 2\Phi\left(-\frac{1}{2}\cdot\frac{\beta}{1+\beta}\cdot\sqrt{T\cdot\frac{(bu-\eta\sigma\rho)^2 + \eta^2\sigma^2\rho_0^2}{b^2\sigma^2\rho_0^2}}\right), \tag{46}$$

where $\Phi(\cdot)$ is the cumulative distribution function of the standard normal distribution. Table 2 displays the detection-error probabilities for all the parameters we apply in Section 6.

Table 2 Detection-error probabilities

β \ T	2	5	10	20
0.3	0.926	0.883	0.836	0.769
0.6	0.880	0.812	0.736	0.634
1	0.841	0.751	0.653	0.526

In line with intuition, the detection-error probability decreases w.r.t the horizon and the ambiguity-aversion level. Anderson et al. (2003) conservatively advocated to use a worst-case scenario for robustness purposes only with a value of β such that $\xi_T(\beta)$ is no less than 10%. As we can see from Table 2, for instance with $T = 5$, our setup leads to a situation where using our worst-case robustness is desirable, since $\xi_5(\beta)$ exceeds 75% for all β no greater than 1.

6.2 Robustness on efficient strategy

If we assume that the AAI believes entirely in the reference model, the preference parameter should be set to $\phi = 0$ which can be obtained by setting $\beta = 0$. Therefore, we do not distinguish between an AAI with $\beta = 0$ and an ANI who ignores the model uncertainty. For problem (RMV), we set the initial wealth at $w_0 = 1$, then one checks that $(0, w_0)$ is in the region $\mathcal{A}_5$. Denote events $R_1 = \{\omega|\tau_2(\omega) \le T\}$ and $R_1^c = \{\omega|\tau_2(\omega) > T\}$; if R_1 occurs, we already commented that the AAI will suspend all her insurance service only to maintain the deterministic common costs, such as salaries for employers, office rental and extra costs at time τ_1, transferring all insurance risk to the reinsurer. Hereafter, we call this behavior quitting the risky market, i.e., she will put all the surplus into the risk-free asset ($l = 0$) and all insurance risk transferred to the reinsurer ($q = 0$).

To analyze the robustness of the effective strategy, we compute the sensitivity of q w.r.t. the ambiguity-aversion (model-uncertainty risk-aversion) parameter β: The derivative of $q(t, \beta)$ w.r.t. β computes explicitly as

$$\begin{aligned}\frac{\partial q(t,\beta)}{\partial \beta} = &- \frac{ub\rho - \eta\sigma}{b^2\sigma\rho_0^2(1+\beta)^2}\Bigg\{W - G_2(t)e^{-r(T-t)} \\ &+ [K - w_0e^{rT} - \frac{\lambda}{r}(e^{rT}-1)]\frac{ATe^{-AT}}{(e^{-AT}-1)^2}e^{-r(T-t)}\Bigg\}\end{aligned} \tag{47}$$

Rearrange it, we obtain that $\frac{\partial q(t,\beta)}{\partial \beta} \ge 0$ if $G_2(t)e^{-r(T-t)} > W \ge G_2(t)e^{-r(T-t)} - [K - W_0e^{rT} - \frac{\lambda}{r}(e^{rT}-1)]\frac{ATe^{-AT}}{(e^{-AT}-1)^2}e^{-r(T-t)}$, and $\frac{\partial q(t,\beta)}{\partial \beta} < 0$ otherwise. In particular, the current wealth should be high enough. If the situation $\frac{\partial q(t,\beta)}{\partial \beta} \ge 0$

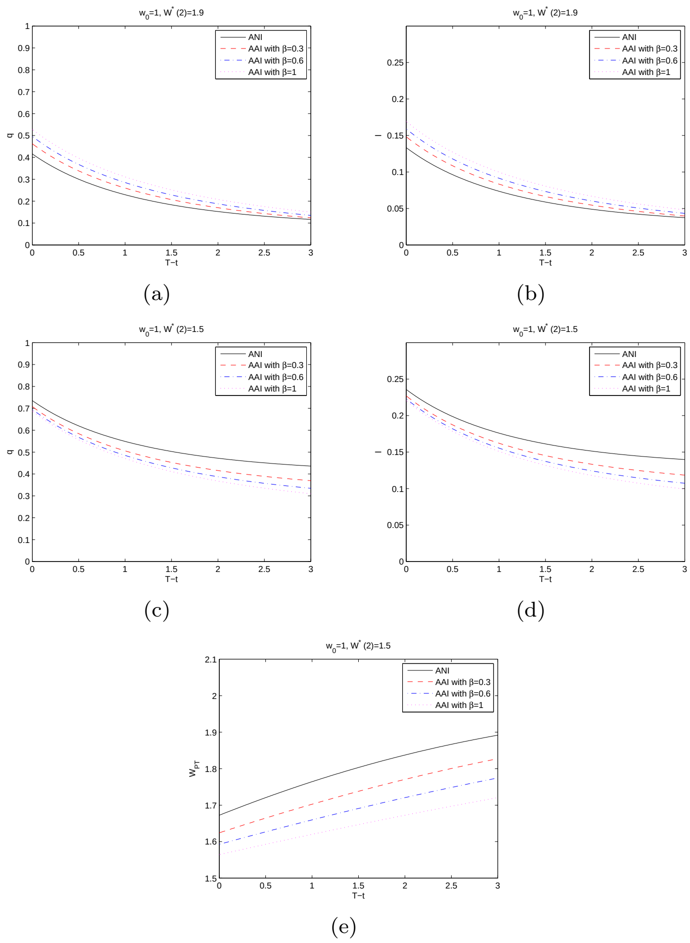

Fig. 1 Model-uncertainty robustness on efficient strategy.

occurs, the insurer with higher ambiguity-aversion level β should keep a larger risk exposure in the reinsurance market as well as in the financial market. In our numerical simulation experiments, we fix the risky asset's Sharpe ratio (risk premium per unit volatility) as $\frac{u}{\sigma} = 0.08$, and perturb σ from 0.25 to 25. We found that $\frac{\partial q(t,\beta)}{\partial \beta} \geq 0$ occurs in all of the one thousand sample paths, whether we are in a mature and stable market (lower u and lower σ), or a market that provides possibilities of very high return in exchange for high risk (very high u and high σ), as might be the case for certain markets in developing economies. Thus, we conclude that the AAI who chooses to accomplish model-uncertainty robustness will increase her risk exposure due to an increase in her preference about ambiguity aversion, when the current wealth is enough high, and will decrease her risk exposure with lower current wealth. Therefore, model-uncertainty robustness plays a remarkably different role compared with the results in the framework of concave utility functions (i.e., Maenhout (2006) and Liu (2010)).

Fixing $t = 2$, Figure 1 illustrates the efficient strategy as a function of the time horizon $T - t$. We set the horizon T to no more than 5 years. It can be stated that both efficient reinsurance and investment decrease as the horizon increases. Figure 1(a) and Figure 1(b) show the efficient mean-variance strategy with $W^*(2) = 1.9$, which could arguably be considered a very high level of current wealth after two years of business. The AAI acquires more insurance business, and holds more risky asset than the ANI. Figure 1(c) and Figure 1(d) show the effective mean-variance strategy for $W^*(2) = 1.5$, a high wealth level after two years, but not high enough to imply a reversal of risk behavior: The ANI acquires larger amounts of reinsurance business and risky asset than AAI. Furthermore, in both cases, neither of the two insurers have the opportunity to quit the risky markets up to $t = 2$.

This phase transition from Figure 1(a) to Figure 1(b) is paradoxical, it does not occur in utility frameworks. We think that it is consistent with the following psychology: For wealthy insurers, taking risk has been beneficial in the (RMV) problem. When they worry about model uncertainty, these insurers may take an even more extreme risk positions based on their positive past experience with high risk exposure. Consequently, how to judge a "good" or "bad" past experience becomes an important question for insurers. In our framework, it depends on the signal provided by the sign of the expression in (48). Figure 1(e) shows the phase transition wealth W_{PT} between the two phases: $\frac{\partial q(t,\beta)}{\partial \beta} < 0$ and $\frac{\partial q(t,\beta)}{\partial \beta} > 0$. We provide the following theorem to judge the attitude of AAI to risk exposure when facing model uncertainty.

Theorem 4 *For all value of $T-t$, for every $W^*(T-t) < W_{PT}(T-t)$, higher "ambiguity-aversion level" β leads to less risky exposure at t. And for every $W^*(T-t) > W_{PT}(T-t)$, higher "ambiguity-aversion level" β leads to more risky exposure at t. In addition, if $W^*(T-t) = W_{PT}(T-t)$, the AAI is locally neutral to the confidence on reference model as measured by β, and will adopt a strategy like an ANI at t.*

Maehout (2006) and Liu (2010) showed that the ambiguity-aversion level β can be understood as an extra risk-aversion coefficient under certain utility functions types (CRRA and recursive preference). They stated higher ambiguity-aversion levels implies a decrease in the amount of risky assets in financial markets. However, the conclusions they proposed are not effective to explain model-uncertainty robustness on the efficient strategy under the mean-variance criterion.

In addition to sensitivity on β, we examine sensitivity of the efficient strategy for the insurer in some of the other parameters. We perturb the values of parameters from 50% to 150% of their base values. Figure 2 indicates the corresponding sensitivities. Figure 2(a) and Figure 2(b) show that the efficient investment strategy is more robust to the insurance parameter λ (difference of insurance-reinsurance premium rates) than the financial market risk-premium

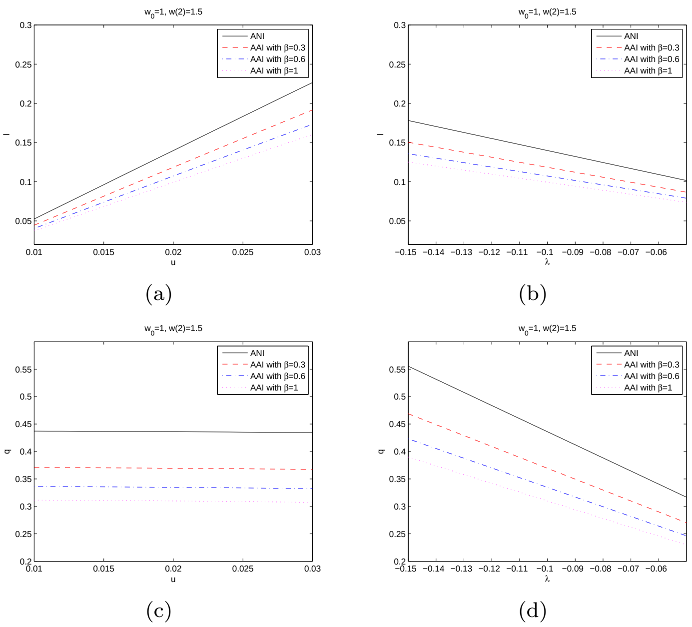

Fig. 2 Sensitivity of the efficient strategy for ANI and AAIs.

parameter u. Figure 2(c) and Figure 2(d) illustrate the efficient reinsurance strategy's strong robustness to the financial market parameter u. In all cases, increasing ones ambiguity-aversion level decreases the sensitivity of efficient strategy: This is consistent with the point of robustness decisions, which is to achieve less sensitivity to model uncertainty the more one worries about model accuracy.

6.3 Robustness on the value function

In this subsection, we aim to clarify the impact of model uncertainty on the value function for problem (RMV). The AAI is an insurer who minimize the $\mathrm{PVar} W^*(T)$ in the worst-case scenario. For problem (RMV), the AAI faces a trade-off with the model uncertainty: She finds a strategy to minimum $\mathrm{PVar}^{Q^*} W(T)$ under the worst-case scenario Q^* while the wealth process $W^*(t)$ guarantees $\mathrm{E}^{Q^*}[W^*(T)] = K$. Hence, $\mathrm{PVar}^{Q^*} W^*(T)$ for AAI may deviate from the traditional $\mathrm{Var} W^*(T)$ for ANI, and be affected by the ambiguity level. To measure the increased spread due to aversion to model uncertainty, compared to the case of model certainty, we define a "discrepancy" function for the standard deviations as follows.

Definition 4 The "discrepancy" function for the AAI, comparing with the ANI, is defined as

$$U := 1 - \left(\frac{J^0_{RMV}(0, w_0)}{J^\beta_{RMV}(0, w_0)} \right)^{\frac{1}{2}}, \tag{48}$$

where $J^\beta_{RMV}(t, w)$ is the value function of problem (RMV) for the AAI with ambiguity level β.

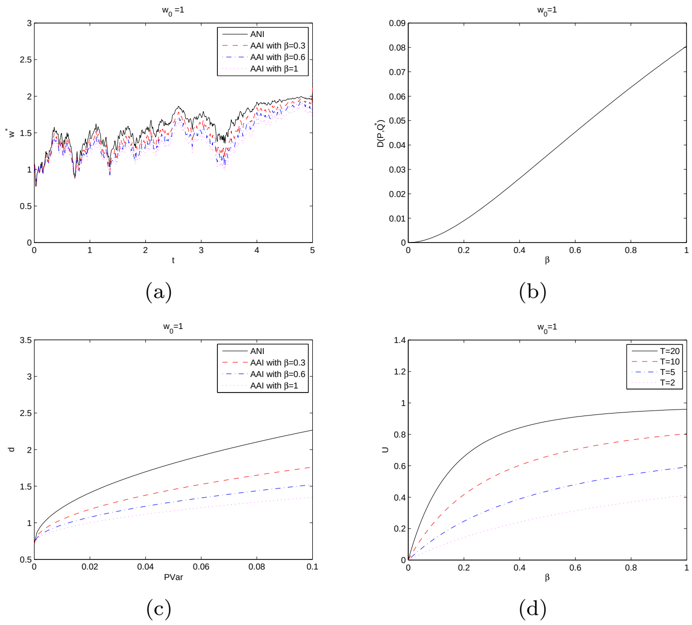

Fig. 3 Model-uncertainty robustness on wealth process and value function.

We also set the initial wealth $w_0 = 1$ and the time horizon $T = 5$, thus, $(0, w_0)$ is in the region $\mathcal{A}_5$. In this case, the drifts under Q^* are given by (46). Thus $D(P, Q^*)$ is given by

$$D(P, Q^*) = \frac{1}{2}({\theta_1^*}^2 + {\theta_2^*}^2) = \frac{\beta^2}{(1+\beta)^2} \cdot \frac{(bu - \eta\sigma\rho)^2 + \eta^2\sigma^2\rho_0^2}{b^2\sigma^2\rho_0^2}. \tag{49}$$

Figure 3(a) presents one sample path of the surplus processes for both the ANI and the AAI. The surplus processes converge to K at maturity $t = 5$. Figure 3(b) shows $D(P, Q^*)$ as a function of ambiguity-aversion level β. With higher β, the AAI loses more confidence in the reference model P, and seeks more model-uncertainty robustness. Figure 3(c) presents the efficient frontier for Problem (RMV). An increase in $\text{PVar}W^*(T)$ occurs when β increases under fixed K, which implies that the AAI would be willing to accept higher variance for the robust optimal strategy when she has less information. Figure 3(d) displays more information about the variance with penalty, which is presented as the discrepancy function from Definition 4, graphed against the variable β. Figure 3(d) also illustrates another significant feature of our robustness model: The "discrepancy" for the AAI in long-horizon is quite large, and alarmingly so for very long horizon: The AAI with $T = 20$ and $\beta = 1$ will generate a discrepancy above 90% of the standard deviation compared to the ANI with the same horizon, which is consistent with the following behavioral interpretation: The very-long-horizon AAI accumulates much more lack of confidence on the reference model than the short-horizon AAI. Consequently, the horizon plays an important role in the behavior of the AAI.

6.4 Quitting probability for the insurer

Finally we turn to the analysis of the quitting probability for the insurer under the mean-variance criterion. According to Theorem 3, the insurer quits the risky market if the pair $(t, W^*(t))$ reaches the boundary of $\mathcal{A}_5$ in the case $ub\rho - \eta\sigma < 0$. Using a simulation, we find that the quitting event $R_1 = \{\omega : \tau_2(\omega) \leq T\}$ only happens in some extremely favorable markets and model uncertainty only impacts the quitting probability in these rare cases.

Table 3 Quitting probability

u \ β	0	2	4	6	8
0.02	0	0	0	0	0
0.2	0	0	0	0	0
0.8	1	0	0	0	0
1.4	1	1	0	0	0
2	1	1	1	1	0.04

Table 3 shows the frequency of event R_1 happening in a five-year period with $\beta \in [0, 8]$ and $u \in [0.02, 2]$, with financial volatility fixed at $\sigma = 0.25$. For "0" in the table, it means some numbers no bigger than 0.001. This table illustrates that the quitting behavior only happens in a extremely high risk-premium markets. In a normal market with $u = 0.02$, it barely happens whatever the ambiguity-aversion level. In addition, the frequency of quitting converges to zero as the ambiguity level increases. In market with $u = 2$, the ANI has an extremely high quitting frequency, while the AAI with $\beta = 8$ has almost no quitting chance even with this very high risk premium.

7 Conclusions

In this paper, we investigated a benchmark problem and a mean-variance problem for an ambiguity-averse insurer (AAI). The surplus process of the insurer is approximated by a diffusion model. The financial market is modeled by a classic Black-Scholes model. At the same time, the AAI lacks full confidence on the pair of economic models. We formulated general robust problems for the insurer under mean-variance and benchmark criteria, and derived the benchmark optimal strategy, the mean-variance efficient strategy and the corresponding value functions based on the connection between the two criteria. Furthermore, we analyzed the model-uncertainty robustness on the mean-variance efficient strategy for the AAI, including the impact of model uncertainty on value function and quitting probability.

The main findings are as follows. (i) A high ambiguity level does not always decrease the risk exposure over the time horizon; it increases the risk exposure if the current wealth is large enough. Therefore, model-uncertainty robustness distinguishes itself from the utilities framework in its possible impacts on the optimal strategy. We think our model provides some explanation of the behavior by which, when facing the model uncertainty, an AAI's attitude to risk exposure switches in the presence of highly favorable past experience. (ii) In normal markets, the behavior in which an insurer is able to quit risky markets, does not occur for the AAI or ANI (ambiguity-neutral insurer). We identify some theoretical markets (very high financial risk premium), where a high ambiguity-aversion level reduces the probability of quitting for the AAI, while an ANI would encounter quitting opportunities much more frequently.

Our mathematical analysis focuses on a dynamic mean-variance problem for an AAI and we have emphasized the effect of model-uncertainty robustness on the mean-variance efficient strategy and quitting probability. In order to extend this study to more realistic risk models, one could attempt to abandon the Black-Scholes model with constant volatility. By considering more flexible models such as jump-diffusion (JD) and stochastic volatility (SV) models, more realistic financial risk situations could be achieved, but explicit expressions for mean-variance efficient strategies for the AAI would be hard to derive explicitly in JD and SV models. Numerical methods could be more useful to analyze the model-uncertainty robustness under these models, to find out how volatility risk and incomplete markets change decisions about model uncertainty aversion. Since our entire framework is based on using the Girsanov theorem, we would still only be able to provide robustness strategies for uncertainty about drift parameters, but the presence of the so-called "vol-vol" parameter in SV models for instance, has the effect of changing models from log-normal to much heavier-tailed mixtures; this could result in a dampening of the phase transition effect we described by which ambiguity-aversion behavior reverses when entering very favorable markets conditions.

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