Kalman Filtering

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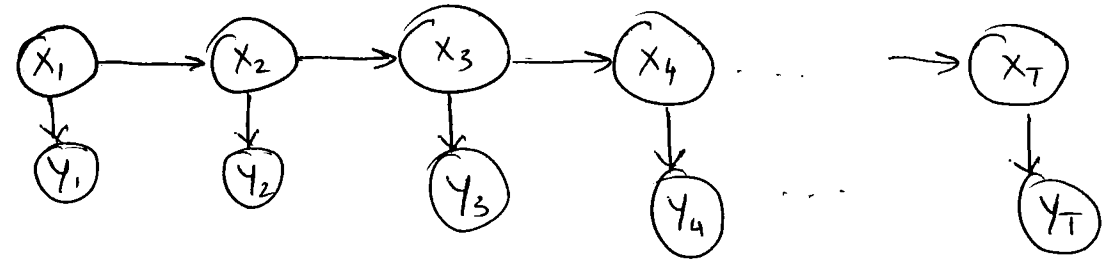

$X_1 \sim N(\mu_0, \Sigma_0)$ Gaussian prior distribution on $X_i \in \mathbb{R}^D$

$X_i = AX_{i-1} + \varepsilon_i$, $\varepsilon_i \sim N(0, \Sigma_E)$ Linear Gaussian dynamics.

$Y_i = BX_i + \xi_i$, $\xi_i \sim N(0, \Sigma_Z)$ Linear Gaussian observations $Y_i \in \mathbb{R}^d$

$p(X_1 \ldots X_T, Y_1 \ldots Y_T)$ $\rightarrow$ A big Gaussian. ($T(D+d)$ - dimensional)

$p(X_1 \ldots X_T \mid Y_1 \ldots Y_T)$ $\rightarrow$ Gaussian conditional

① $p(X_i \mid Y_1 \ldots Y_T)$ $\rightarrow$ Gaussian marginal.

Can't calculate directly because of large matrices involved. Is $O(T^3)$.

Calculating Equation ① is the <u>smoothing problem</u>.

$$P(X_i \mid Y_1 \ldots Y_T) = \frac{P(X_i, Y_1 \ldots Y_T)}{P(Y_1 \ldots Y_T)}$$

$\uparrow$ A function of X_i (a density in fact)

$P(Y_1 \ldots Y_T) \leftarrow$ A constant that doesn't depend on X_i, discard it.

$$\propto P(X_i, Y_1 \ldots Y_T)$$

$$= P(X_i, Y_1 \ldots Y_i)\, P(Y_{i+1} \ldots Y_T \mid X_i, Y_1 \ldots Y_i)$$

$$= P(X_i, Y_1 \ldots Y_i)\, P(Y_{i+1} \ldots Y_T \mid X_i) \qquad \text{(Markov Property)}$$

$$= \alpha_i(X_i)\, \beta_i(X_i) \leftarrow$$ Backward message, information from the future.

$\uparrow$ Forward message: Information from the past & present

Note: $\alpha_i(X_i) = P(X_i, Y_1 \ldots Y_i) \propto P(X_i \mid Y_1 \ldots Y_i)$.
Calculating this is the filtering problem.

$$\alpha_i(X_i) = P(X_i, Y_1 \ldots Y_i) = \int P(X_i, X_{i-1}, Y_1 \ldots Y_i)\, dX_{i-1}$$

$$= \int P(X_{i-1}, Y_1 \ldots Y_{i-1})\, P(X_i \mid X_{i-1}, Y_1 \ldots Y_{i-1})\, P(Y_i \mid X_i, X_{i-1}, Y_1 \ldots Y_{i-1})\, dX_{i-1}$$

$$= \int \alpha_{i-1}(X_{i-1})\, P(X_i \mid X_{i-1})\, P(Y_i \mid X_i)\, dX_{i-1} \qquad \text{(Markov)}.$$

$\uparrow$ New filtering estimate; Previous filtering estimate; Prediction; Update with new data

We thus have a recursion relating α_i to α_{i-1}. Given α_{i-1} we need to solve the above integral over X_{i-1} (and NOT $\{X_1 \ldots X_{i-1}\}$)

Convince yourself that the integral is solvable (even if you might not know the exact solution). In particular, convince yourself that the integrand is of the form $Z_{i-1} \exp\left(-\frac{1}{2}(X_{i-1} - \mu_{i-1})^T \Sigma_{i-1}^{-1} (X_{i-1} - \mu_{i-1})\right)$, for some parameters $Z_{i-1}, \mu_{i-1}, \Sigma_{i-1}$. This does not depend on T.

Now, we can successively calculate $\alpha_1, \alpha_2 \ldots \alpha_T$. Overall, this is $O(T)$

We have solved the filtering problem, now on to the smoothing problem.
How do we calculate $\beta_i(X_i)$?

$$\beta_i(X_i) = P(Y_{i+1} \ldots Y_T \mid X_i) = \int P(Y_{i+1}, \ldots Y_T, X_{i+1} \mid X_i)\, dX_{i+1}$$

$$= \int p(X_{i+1} \mid X_i)\, p(Y_{i+1} \mid X_{i+1}, X_i)\, p(Y_{i+1}, \ldots Y_T \mid X_i, X_{i+1}, Y_{i+1})\, dX_{i+1}$$

$$= \int p(X_{i+1} \mid X_i)\, p(Y_{i+1} \mid X_{i+1})\, \beta_{i+1}(X_{i+1})\, dX_{i+1}. \qquad (1)$$

We now have a backward recursion for β_i from β_{i+1}.

$$\text{Also } \beta_{T-1}(X_{T-1}) = P(Y_T \mid X_{T-1}) = \int p(Y_T, X_T \mid X_{T-1})\, dX_T$$

$$= \int p(\cancel{Y_T}\, X_T \mid X_{T-1})\, p(Y_T \mid X_T)\, dX_T. \qquad (2)$$

Again, convince yourself that eqs (1) & (2) are solvable.
We can thus successively calculate $\beta_{T-1}, \beta_{T-2} \ldots \beta_1$.
This is the backward message passing stage.

Now, $P(X_i \mid Y_1 \ldots Y_T) \propto \alpha_i(X_i)\, \beta_i(X_i)$.

We have both terms, and can also calculate the normalization term.

Extra:- Can you calculate $P(X_i, X_{i+1} \mid Y_1 \ldots Y_T)$ as a function of the α's, β's and the system parameters?