Lecture 6: Complexity, data-structures and sorting

STAT 545: Intro. to Computational Statistics

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Let x and y be $N \times 1$ vectors

Let A and B be $N \times N$ and $N \times M$ matrices

How many additions and multiplications to calculate:

- $x^\top y$
- Ax
- AB
- A^{-1}

Big-O notation

The big-O notation provides an asymptotic upper bound:

$$O(g(N)) = \{f : \exists c, N_0 > 0 \; s.t. \; f(N) \leq cg(N) \; \forall N > N_0\}$$

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So is matrix multiplication $O(N^3)$? Yes, but: it's also $O(N^{2.38})$!

Conjecture: matrix multiplication is actually $O(N^2)$.

Consider a set of N numbers. We want to sort them in decreasing order. What is the complexity?

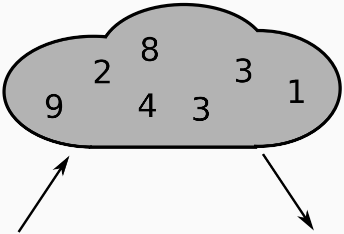
Naïve algorithm:

- Find smallest number. Cost? $O(N)$
- Find next smallest number. Cost? $O(N)$
- ...

Overall cost? $O(N^2)$
Can we do better?

Priority queue

A 'bag' with three commands: `Insert`, `FindMax` and `RemoveMax`.



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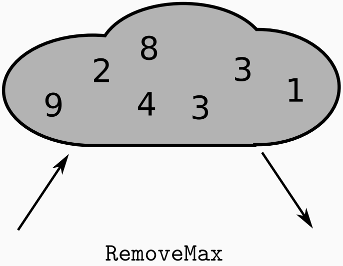

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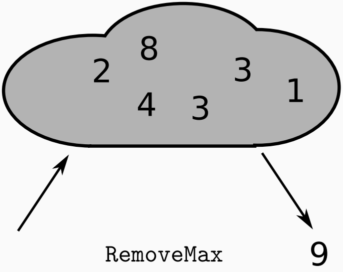

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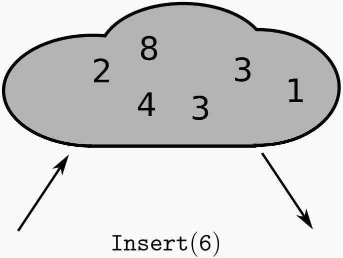

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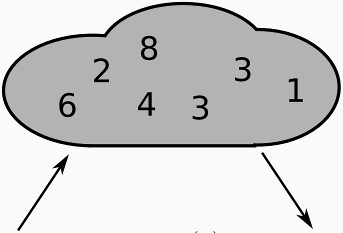
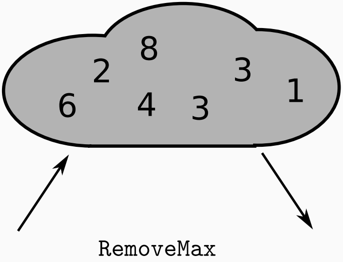


`Insert(6)`

A ‘bag’ with three commands: `Insert`, `FindMax` and `RemoveMax`.



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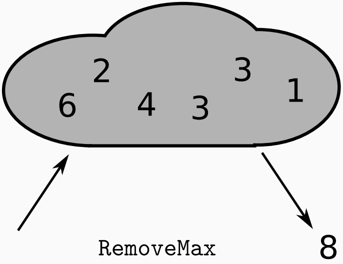

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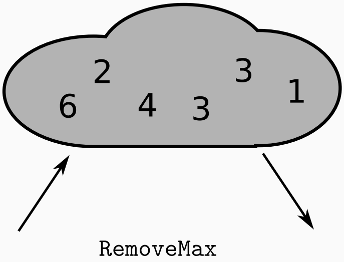

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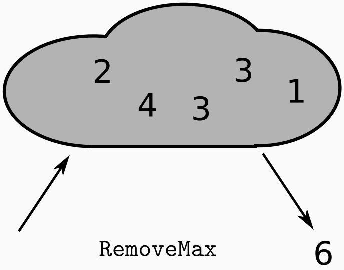

Priority queue

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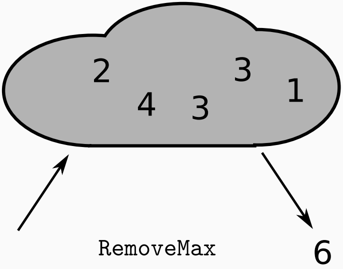


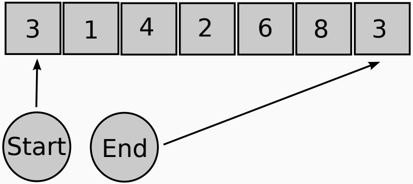
Sorting, clustering, discrete-event simulation, queuing systems

How do we implement this?

Priority queue

Naive approach 1: an unsorted array:

$$(3, 1, 4, 2, 6, 8, 3)$$



Priority queue

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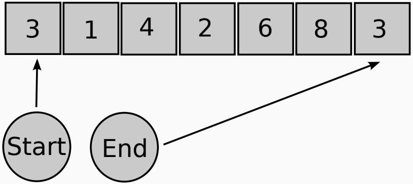


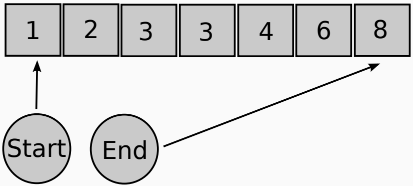
What is the cost of `Insert`? $O(1)$

What is the cost of `FindMax`? $O(N)$

What is the cost of `RemoveMax` (assume we've already found the maximum)? $O(1)$

Naive approach 2: a sorted array:

$$(1, 2, 3, 3, 4, 6, 8)$$



Naive approach 2: a sorted array:

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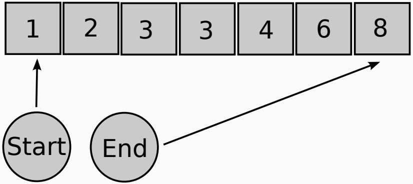


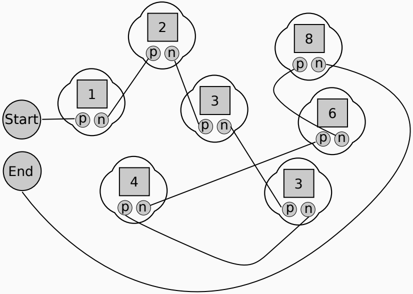
Cost of `FindMax`? $O(1)$

Cost of `RemoveMax`? $O(1)$

Cost of `Insert.FindPosition`? $O(\log(N))$ (binary search)

Cost of `Insert.Insert`? $O(N)$

Naive approach 3: a sorted linked-list:



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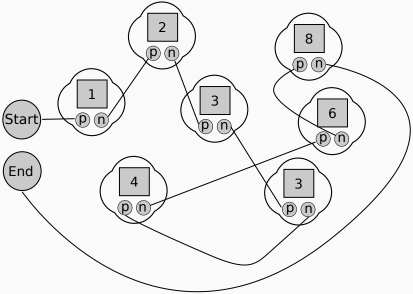

What is the cost of `FindMax`? $O(1)$

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What is the cost of `Insert`? $O(N)$

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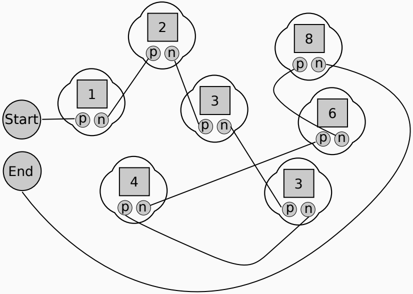


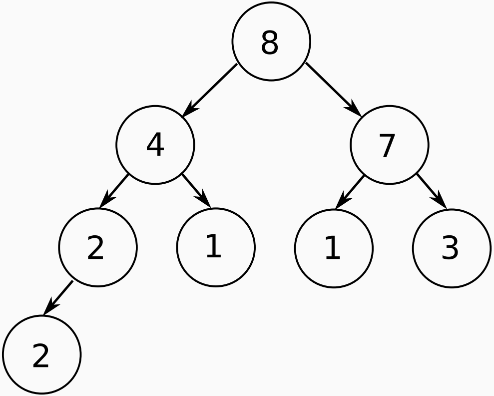
What is the cost of `FindMax`? $O(1)$

What is the cost of `RemoveMax`? $O(1)$

What is the cost of `Insert`? $O(N)$

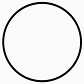
Each approach solves one problem, but makes another operation $\log(N)$. Can we do better?

We need a more complicated data-structure: a Heap.



For a precise definition, see: `http://pages.cs.wisc.edu/~vernon/cs367/notes/11.PRIORITY-Q.html`

Heaps



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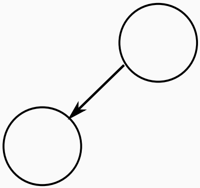

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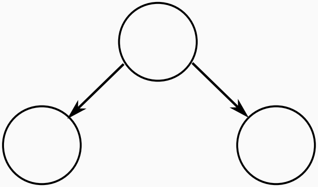

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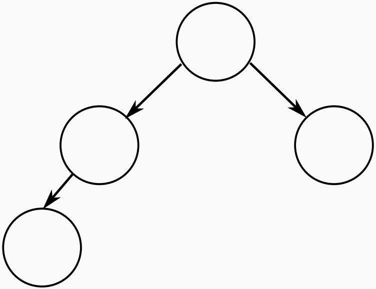

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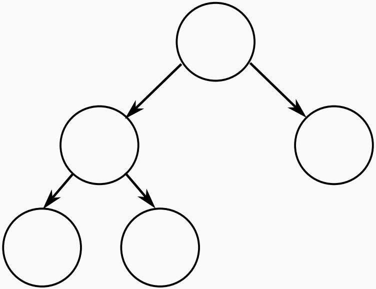

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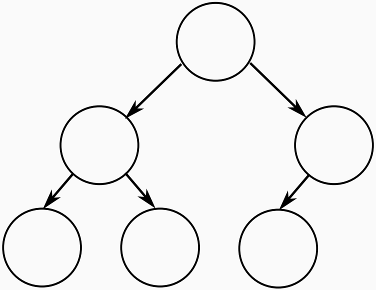

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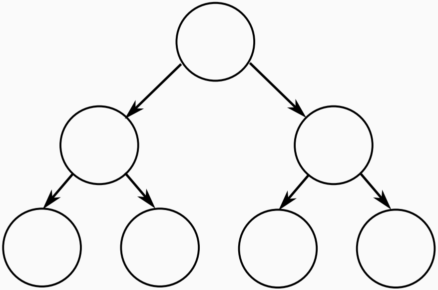

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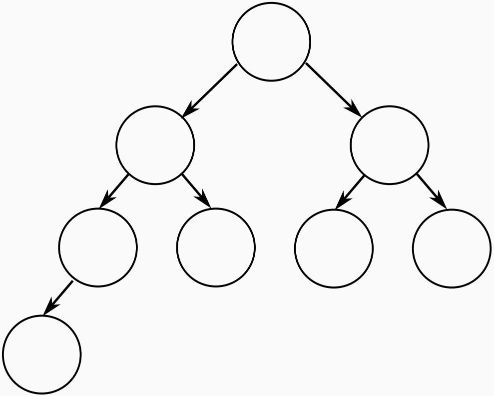

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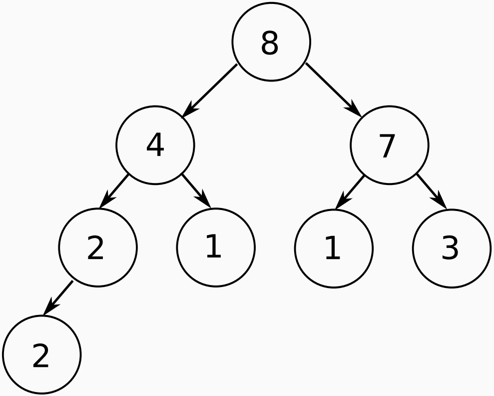

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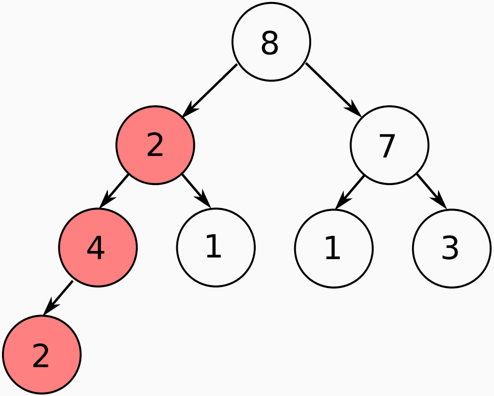

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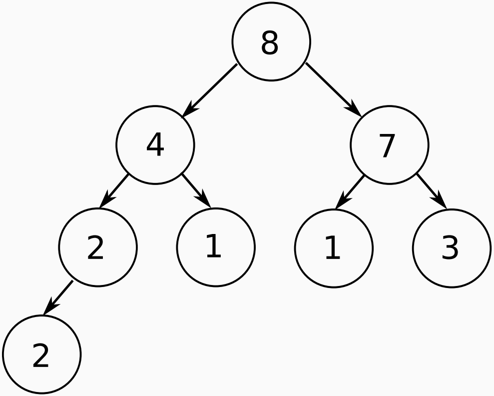
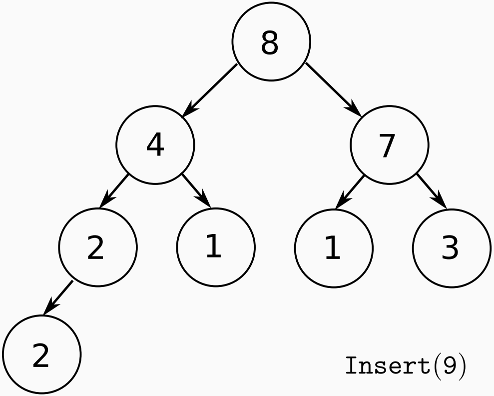
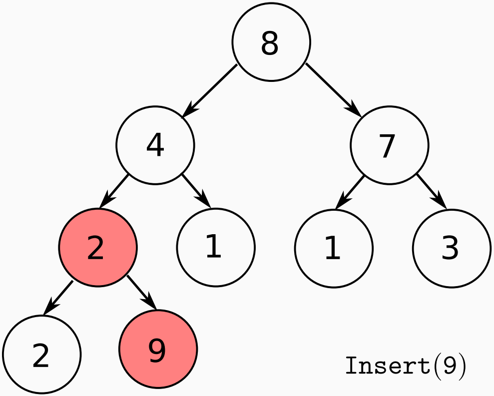
HEAPS: `Insert`





Insert(9)



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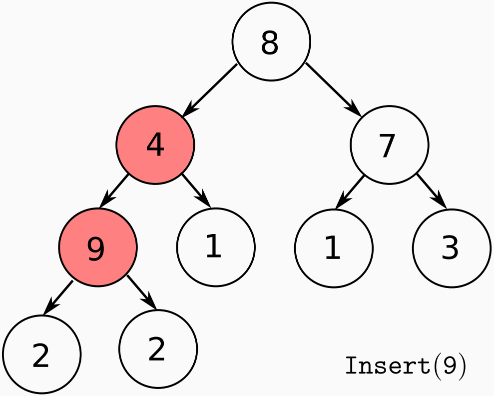

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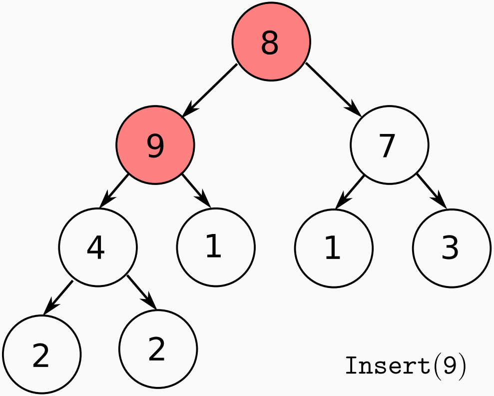

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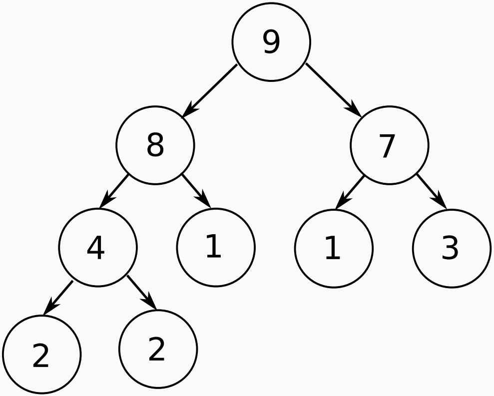
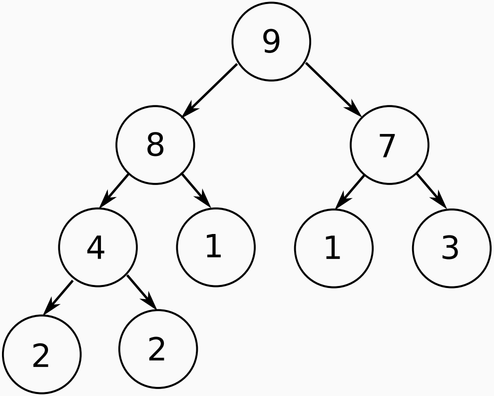
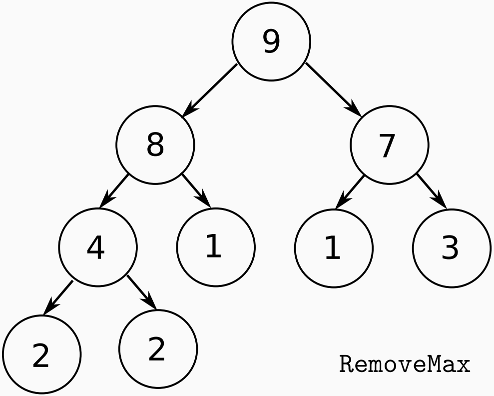

9
8
7
4
1
1
3
2
2

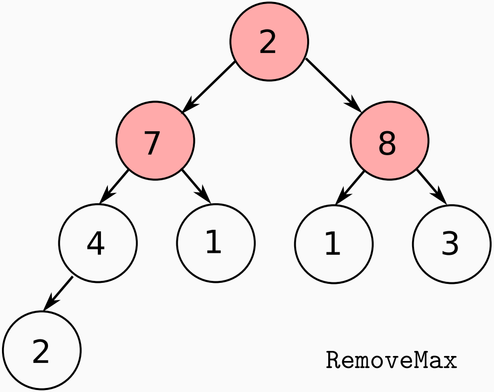
HEAPS: Insert



Cost? $O(\log(N))$



RemoveMax



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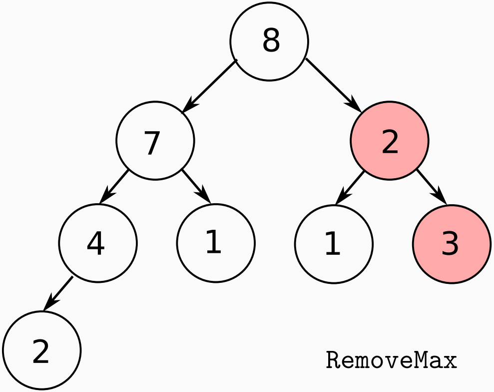
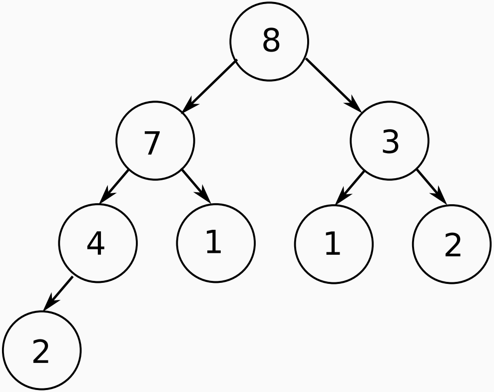


Swap with larger child


8
7
3
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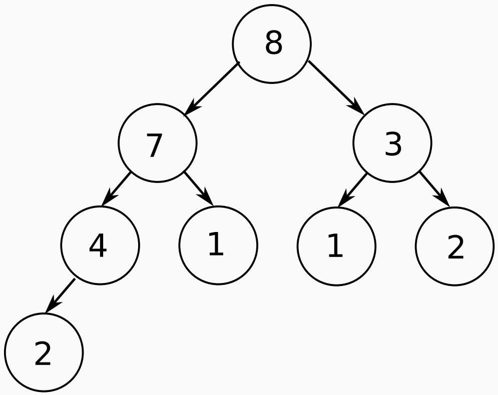
2
2

HEAPS: RemoveMax



Cost? $O(\log(N))$

Heapsort

Consider a set of N numbers. Want to sort in decreasing order.

Grow a priority queue, sequentially adding elements

- Cost of each step? $O(\log(N))$
- Overall cost? $O(N \log(N))$

Sequentially remove the maximum element

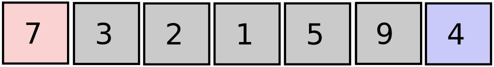
- Cost of each step? $O(\log(N))$
- Overall cost? $O(N \log(N))$

Cost of overall algorithm? $O(N \log(N))$

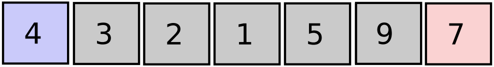
Quicksort

7	3	2	1	5	9	4

Quicksort

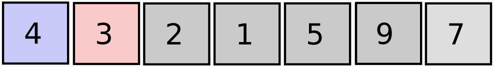


- Pivot: #7
- Start: #1
- End: #6



- Pivot: #1
- Start: #2
- End: #6

Quicksort



- Pivot: #1
- Start: #2
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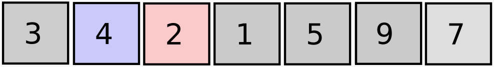
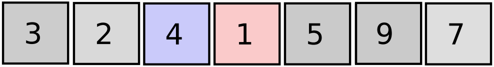
Quicksort



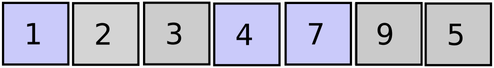
- Pivot: #2
- Start: #3
- End: #6



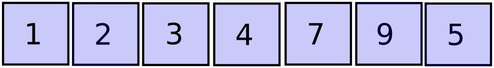
- Pivot: #3
- Start: #4
- End: #6

Recurse for each half

Recurse for each half



Recurse for each half



At the end, we have a sorted list

Analysis of quicksort

Analysis is a bit harder

What is the worst-case runtime?

What is the best-case runtime?

Average run-time is $\Theta(n \log n)$

Average with respect to what?

Randomized algorithms

Final (informal) comments

Class **P**: Problems of polynomial complexity. Let $T(n)$ be running-time for input size n. Then there is a k such that:

$$T(n) = O(n^k)$$

Class **E**: Problems of exponential complexity.

$$T(n) = \exp(O(n))$$

Class **NP**: Problems of where proposed solution can we verified in polynomial time. E.g. graph isomorphism
Class **NP**-hard: at least as hard as the hardest problems in **NP** (halting problem)
Class **NP**-complete: Hardest problems in **NP** (i.e. problems in both **NP** and **NP**-hard). E.g. travelling salesman.

A million dollar question (literally)