The Dose Problem

An important interest is to predict the value of the explanatory variable in a logistic linear model for a target probability. Assume that a simple logistic linear model is given by

$$\log \frac{p_i}{1-p_i} = \beta_0 + x_i\beta_1$$

for all $i \in \{1, \cdots, I\}$. Denote the estimates of β_0 and β_1 as $\hat{\beta}_0$ and $\hat{\beta}_1$, and their variance covariance estimator as

$$\widehat{\mathrm{cov}} \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \end{pmatrix} = \begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{pmatrix},$$

where $v_{01} = v_{10}$.

Suppose we want to find a value of x such that the predicted value of p is p_0. Then, we have

$$\log \frac{p_0}{1-p_0} = \hat{\beta}_0 + \hat{\beta}_1 x \Rightarrow \hat{x} = \frac{1}{\hat{\beta}_1}\left(\log \frac{p_0}{1-p_0} - \hat{\beta}_0\right).$$

The variance of $\hat{x}$ is derived by the Delta theorem. By

$$\begin{pmatrix} \frac{\partial \hat{x}}{\partial \beta_0} & \frac{\partial \hat{x}}{\partial \beta_1} \end{pmatrix} = \begin{pmatrix} -\frac{1}{\hat{\beta}_1} & -\frac{\log[p_0/(1-p_0)]-\hat{\beta}_0}{\hat{\beta}_1^2} \end{pmatrix},$$

we obtain the variance of $\hat{x}$ as

$$\begin{aligned} \widehat{\mathrm{var}}(\hat{x}) = & \begin{pmatrix} \frac{\partial \hat{x}}{\partial \beta_0} & \frac{\partial \hat{x}}{\partial \beta_1} \end{pmatrix} \begin{pmatrix} v_{00} & v_{01} \\ v_{10} & v_{11} \end{pmatrix} \begin{pmatrix} \frac{\partial \hat{x}}{\partial \beta_0} \\ \frac{\partial \hat{x}}{\partial \beta_1} \end{pmatrix} \\ = & \frac{v_{00}}{\hat{\beta}_1^2} + \frac{2v_{01}\{\log[p_0/(1-p_0)] - \hat{\beta}_0\}}{\hat{\beta}_1^3} + \frac{v_{11}\{\log[p_0/(1-p_0)] - \hat{\beta}_0\}^2}{\hat{\beta}_1^2}. \end{aligned}$$

Then, we can compute the $1-\alpha$ level confidence interval for the dose as

$$\hat{x} \pm z_{\alpha/2}\sqrt{\widehat{\mathrm{var}}(\hat{x})}.$$

Example. In the `bliss` example, then we have $\hat{\beta}_0 = -2.3238$, $\hat{\beta}_1 = 1.1619$, $v_{00} = 0.1746$, $v_{01} = v_{10} = -0.0658$, and $v_{11} = 0.0329$. If we choose $p_0 = 0.9$, then

$$\log \frac{0.9}{0.1} = \hat{\beta}_0 + \hat{\beta}_1 x \Rightarrow \hat{x} = \frac{2.197 + 2.3238}{1.1619} = 3.8909.$$

By

$$\begin{pmatrix} \frac{\partial \hat{x}}{\partial \beta_0} & \frac{\partial \hat{x}}{\partial \beta_1} \end{pmatrix} = (-0.8607, -3.3489),$$

we have

$$\widehat{\mathrm{var}}(\hat{x}) = 0.8607^2(0.1746) + 2(0.8607)(3.3489)(-0.0658) + 3.3489^2(0.0329) = 0.1190.$$

Then, the 95% confidence interval for x when $p_0 = 0.9$ is

$$3.8909 \pm 1.96\sqrt{0.1190} = [3.6771, 4.1047].$$