1. 4.1.2.

Solution:

(a) Omitted. Do the plot by yourself.
(b) Using the method in my notes, we have the MLE \( \hat{\mu} = \bar{X} \) and \( \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^2 / m \). Using the data, we have the maximum likelihood estimates of \( \mu, \sigma^2, \sigma \), and \( \mu/\sigma \) are \( \hat{\mu} = 201, \hat{\sigma}^2 = 293.92, \hat{\sigma} = \sqrt{293.92} = 17.14 \), and \( \hat{\mu}/\hat{\sigma} = 11.72 \), respectively. Then, you need to plot the curve of the PDF of \( N(201, 17.14^2) \) in your histogram.
(c) Let \( X \) be the count of values over 215. We have \( x = 7 \). Thus, the MLE of \( p \) is \( \hat{p} = 7/26 = 0.2692 \).
(d) Suppose that \( Y \sim N(201, 17.14^2) \). The MLE of the probability is
\[
P(Y > 215) = 1 - \Phi\left(\frac{215 - 201}{17.14}\right) = 1 - \Phi(0.82) = 0.2061.
\]

2. 4.1.3.

Solution:

(a) By the method in my lecture, the MLE of \( \theta \) is \( \hat{\theta} = \bar{X} \). Since \( E(\hat{\theta}) = E(\bar{X}) = \theta \), it is an unbiased estimator of \( \theta \).
(b) Based on the data, we have \( \hat{\theta} = 9.5 \). We expect 9.5 customers entering the store between 9:00am and 10:00am. It can be interpreted as the long-term average.

3. 4.1.4.

Solution: This problem asks you to write down the detail of steps in maximum likelihood estimation. You need to write down the likelihood function, the log-likelihood function and its derivative, and the solution of the estimating equation. Please read example the Poisson example in my notes.

4. 4.1.6.

Solution: Note that \( X_1, \ldots, X_n \) are iid. We have \( I_j(X_i), \ldots, I_j(X_n) \) are iid. We have
\[
V\left[\frac{1}{n} \sum_{i=1}^{n} I_j(X_i)\right] = \frac{1}{n} V[I_j(X_j)].
\]
We next compute \( V[I_j(X_i)] \). Based on the PMF \( P(X_i = a_j) = p_j \) and \( P(X_i \neq a_j) = 1 - p_j \). We have \( V[I_j(X_j)] = p_j(1 - p_j) \). Thus,
\[
V\left[\frac{1}{n} \sum_{i=1}^{n} V[I_j(X_i)]\right] = \frac{p_j(1 - p_j)}{n}.
\]
The mgf is that of binomial distribution. You can use the formula. It is
\[
M(t) = [(1 - p_j) + p_j e^t]^n.
\]
Note: If the student does not do MGF, it is fine.

5. 4.1.8.

Solution: The MLE is \( \hat{\theta} = \bar{X} \). Based on the data, we obtain the maximum likelihood estimate of \( \theta \) is \( \hat{x} = 2.133 \). Using \( Poiss(2.13) \), we obtain the MLE of the PMF as
6. 4.2.1.

**Solution:** We have $\bar{x} = 81.2$ and $s^2 = 26.5$. We use $t$-confidence intervals given by $\bar{x} \pm t_{\alpha/2, s^2/n^{1/2}}$. Using $\alpha = 0.10$, $0.05$, and $0.01$, we obtain the 90% confidence interval for $\mu$ as $[79.21, 83.19]$, the 95% confidence interval for $\mu$ as $[78.79, 83.61]$, the 99% confidence interval for $\mu$ as $[77.91, 84.49]$. Their lengths are 3.98, 4.82, and 6.59, respectively.

7. 4.2.6.

**Solution:** The 90% confidence interval for $\mu$ is $X \pm t_{0.05, 16} \frac{\sqrt{5.76}}{\sqrt{17}} \approx 4.7 \pm 1.746 \frac{\sqrt{5.76}}{\sqrt{17}} = [3.684, 5.716]$.

9. 4.2.8.

**Solution:** The length of the 95% confidence interval for $\mu$ is $2z_{0.025} \sigma / \sqrt{n}$. We need it to be less than 1. Thus, we have

$$2z_{0.025} \frac{\sigma}{\sqrt{n}} \leq 1 \Rightarrow n \geq \frac{4z_{0.025}^2 \sigma^2}{1^2} = \frac{4(1.96)^2(10)}{1} = 153.6.$$

Thus, we choose $n = 154$.

10. 4.2.10.

**Solution:** It is clear that $E(\bar{X} - X_{n+1}) = 0$ and

$$V(\bar{X} - X_{n+1}) = V(\bar{X}) + V(X_{n+1}) = \frac{n+1}{n} \sigma^2.$$ 

and

$$(n-1)S^2 \sim \sigma^2 \chi^2_{n-1}.$$ 

Therefore,

$$T = \frac{(\bar{X} - X_{n+1})/\sqrt{(n+1)\sigma^2/n}}{S/\sigma} \sim t_{n-1}$$

which implies

$$c = \sqrt{\frac{n}{n+1}}.$$ 

If $k = 8$, then

$$P(\bar{X} - kS < X_9 < \bar{X} + kS) = 0.80 \Rightarrow P(\frac{\bar{X} - X_9}{S} < k) = 0.80$$

$$= P(\sqrt{8} \frac{\bar{X} - X_9}{S} < k \sqrt{8}) = 0.80$$

$$= k \sqrt{\frac{8}{9}} = t_{0.18}$$

$$= k = t_{0.18} \times \sqrt{\frac{8}{9}} = 1.482.$$