There are totally 35 points in the exam. The exam is take home, open-book and open-notes. Please submit your solutions of the exam to the BrightSpace. Please include your name, student ID number, and your section in your file.

NAME: ________________________________
ID: ________________________________
Section 1: __ 12:30-1:20pm, MWF
Section 2: __ 1:30-3:20pm, WMF
1. (15 points). Fill in the blanks (1 point each).

(a) (3 points). Suppose a bag has 6 red balls and 8 yellow balls. Randomly choose 4 balls. The probability of exactly two red balls is \( \binom{6}{2} \binom{8}{2} \). The probability of at most one red ball is \( \binom{6}{1} \binom{8}{3} + \binom{6}{0} \binom{8}{4} \). The probability of at least 3 red balls is \( \binom{6}{3} \binom{8}{1} + \binom{6}{4} \binom{8}{0} \).

(b) (3 points). Suppose a box has 3 bags. The first bag has 5 yellow balls and 5 red balls. The second bag has 4 yellow balls and 16 red balls. The third bag has 2 yellow balls and 3 red balls. Assume one first chooses one bag and then chooses a ball from the chosen bag. The conditional probability that the chosen ball is red if the chosen bag is the first bag is \( \frac{5}{10} \). The probability that the chosen ball is red is \( \frac{3}{15} \). Assume the chosen ball is red. The probability that the chosen bag is the first bag is \( \frac{5}{10} \).

(c) (3 points). Assume \( A, B, \) and \( C \) are independent with \( P(A) = 0.5, P(B) = 0.6 \) and \( P(C) = 0.8 \). Then, \( P(A \cap B \cap C) = \frac{5}{10} \cdot \frac{6}{10} \cdot \frac{8}{10} \), \( P(A \cup B | C) = \frac{P(A \cap C) + P(B \cap C)}{P(C)} \), and \( P(A \cap \overline{C}) = P(A) \cdot P(\overline{C}) \).

(d) (3 points). Assume a person is shooting a target. The probability for the person to hit the target is 0.6. Assume the happening is independent and the person has 8 bullets. Let \( X \) be the total number of bullets hitting the target. Then, \( P(X \leq 6) = \frac{5}{10} \cdot \frac{6}{10} \), \( P(4 \leq X \leq 6) = \frac{5}{10} \cdot \frac{6}{10} \cdot \frac{8}{10} \), and \( V(X) = \frac{5}{10} \cdot \frac{6}{10} \cdot \frac{8}{10} \).

(e) (3 points). Suppose \( X \sim N(3, 16) \). Then, \( E(X) = 3 \), \( V(X) = 16 \), and \( P(0 \leq X \leq 5) = \frac{5}{10} \cdot \frac{6}{10} \).

Answer: (a) \( \binom{6}{2} \binom{8}{2} = 0.4196 \), \( \binom{6}{1} \binom{8}{3} + \binom{6}{0} \binom{8}{4} = 0.4056 \), and \( \binom{6}{3} \binom{8}{1} + \binom{6}{4} \binom{8}{0} = 0.1748 \). (b) 0.5; 0.6333; 0.2632. (c) 0.24; 0.8; 0.1. (d) 0.8936; 0.7200; 1.92. (e) 3, 16, 0.1241.
2. (8 points). Suppose the joint PMF of $X$ and $Y$ is

<table>
<thead>
<tr>
<th>$X$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0.04</td>
<td>0.05</td>
<td>0.09</td>
<td>0.02</td>
</tr>
<tr>
<td>1</td>
<td>0.08</td>
<td>0.10</td>
<td>0.08</td>
<td>0.04</td>
</tr>
<tr>
<td>2</td>
<td>0.05</td>
<td>0.05</td>
<td>0.07</td>
<td>0.03</td>
</tr>
<tr>
<td>3</td>
<td>0.07</td>
<td>0.05</td>
<td>0.09</td>
<td>0.09</td>
</tr>
</tbody>
</table>

(a) (2 points). Compute $P(|X - Y| = 1)$.

$$P(|X - Y| = 1) = 0.05 + 0.08 + 0.03 + 0.08 + 0.05 + 0.09 = 0.38.$$ 

(b) (2 points). Compute the marginal PMFs of $X$ and $Y$, respectively.

The marginal PMF of $X$ is

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_X(x)$</td>
<td>0.2</td>
<td>0.3</td>
<td>0.2</td>
<td>0.3</td>
</tr>
</tbody>
</table>

and the marginal PMF of $Y$ is

<table>
<thead>
<tr>
<th>$y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_Y(y)$</td>
<td>0.24</td>
<td>0.25</td>
<td>0.33</td>
<td>0.18</td>
</tr>
</tbody>
</table>

(c) (2 points). Compute $Cov(X, Y)$ and $Corr(X, Y)$.

Based on the marginal PMFs, we have $E(X) = 1.6$, $V(X) = 1.24$, $E(Y) = 1.45$ and $V(Y) = 1.0875$. Based on the joint PMF, we have $E(XY) = 2.44$. Thus, Thus,

$$Cov(X, Y) = 2.44 - 1.6 \times 1.45 = 0.12$$

and

$$Corr(X, Y) = \frac{0.12}{\sqrt{1.24 \times 1.0875}} = 0.1033.$$ 

(d) (2 points). Compute $E(|X - Y|)$.

$$E(|X - Y|) = (0.05 + 0.08 + 0.03 + 0.08 + 0.05 + 0.09) \times 1$$

$$+ (0.09 + 0.04 + 0.05 + 0.05) \times 2 + (0.02 + 0.07) \times 3$$

$$= 1.11.$$
3. (6 points). Suppose $X_1, \ldots, X_n$ are identically independently distributed (iid) with common expected value $\mu$ and variance $\sigma^2$. Let $\bar{X} = \sum_{i=1}^{n} X_i/n$. Use the Central Limit Theorem.

(a) (2 points). Let $X_i$ be numbers of inserts in a bottle with $\mu = 15$ and $\sigma = 6$. Compute $P(14 \leq \bar{X} \leq 16)$ when $n = 60$. (Hint: you must consider the 0.5 shift).

By the CLT, we have

$$\bar{X} \sim_{\text{approx}} N(15, \frac{6^2}{60}) = N(15, 0.6).$$

Thus,

$$P(14 \leq \bar{X} \leq 16) = P(\sum_{i=1}^{60} X_i \leq 960)$$

$$= P(\sum_{i=1}^{40} X_i \leq 960.5)$$

$$= P(13.99 \leq X \leq 16.01)$$

$$\approx \Phi\left(\frac{16.01 - 15}{\sqrt{0.6}}\right) - \Phi\left(\frac{13.99 - 15}{\sqrt{0.6}}\right)$$

$$= \Phi(1.30) - \Phi(-1.30)$$

$$= 0.8064.$$

(b) (2 points). Let $X_i$ be weight of a new born baby with $\mu = 3500g$ and $\sigma = 500$. Compute $P(3450 \leq \bar{X} \leq 3550)$ when $n = 200$

By the CLT, we have

$$\bar{X} \sim_{\text{approx}} N(3500, \frac{500^2}{200}) = N(3500, 1250).$$

Thus,

$$P(3450 \leq \bar{X} \leq 3500) \approx \Phi\left(\frac{3550 - 3500}{\sqrt{1250}}\right) - \Phi\left(\frac{3450 - 3500}{\sqrt{1250}}\right) = \Phi(1.41) - \Phi(-1.41) = 0.8414.$$

(c) (2 points). Suppose the PMF of $X_i$ is given below. Compute $P(|\bar{X} - 1| \geq 0.01)$ for $n = 10^3$, $n = 10^4$, and $n = 10^5$, respectively.

<table>
<thead>
<tr>
<th>x</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.25</td>
<td>0.50</td>
<td>0.25</td>
</tr>
</tbody>
</table>

In this problem, we have $\mu = 1$ and $\sigma^2 = 0.5$. Thus, we have

$$\bar{X} \sim_{\text{approx}} N(1, \frac{0.5}{n}) = N(1, 0.5/n).$$
Then, we have

\[
P(|\bar{X} - 1| \geq 0.01) = 1 - \Phi(0.99 \leq \bar{X} \leq 1.01)
\]

\[
= 1 - \Phi\left(\frac{0.01}{\sqrt{0.5/n}}\right) - \Phi\left(-\frac{0.01}{\sqrt{0.5/n}}\right)
\]

\[
= \begin{cases} 
1 - [\Phi(0.44) - \Phi(-0.44)] = 0.3274 & \text{when } n = 1000 \\
1 - [\Phi(1.41) - \Phi(-1.41)] = 0.0786 & \text{when } n = 10000 \\
1 - [\Phi(4.47) - \Phi(-4.47)] = 0 & \text{when } n = 100000
\end{cases}
\]
4. (6 points). Compute the following probabilities by using properties of the linear combination of normal distribution. Suppose $X_1, X_2 \sim N(3, 9), X_3 \sim N(4, 10), \text{and } X_4 \sim N(2, 5)$. Assume independence.

(a) (2 points). Compute $P(1.1X_1 + 1.3X_2 + 1.2X_3 \leq 2X_4 + 16)$.

Let $Y = 1.1X_1 + 1.3X_2 + 1.2X_3 - 2X_4$. Then,

$$E(Y) = 1.1E(X_1) + 1.3E(X_2) + 1.2E(X_3) - 2E(X_4) = 8$$

and

$$V(Y) = 1.1^2V(X_1) + 1.3^2V(X_2) + 1.2^2V(X_3) + 2^2V(X_4) = 60.5.$$  

Thus,

$$P(1.1X_1 + 1.3X_2 + 1.2X_3 \leq 2X_4 + 16) = P(Y < 16) = P(N(8, 60.5) \leq 16) = \Phi\left(\frac{16 - 8}{\sqrt{60.5}}\right) = \Phi(1.03) = 0.8485.$$  

(b) (2 points). Compute $P(|1.4X_1 + 1.1X_2 + 1.9X_3 - 13| \leq 5)$.

Let $Y = 1.4X_1 + 1.1X_2 + 1.9X_3 - 10$. Then

$$E(Y) = 1.4E(X_1) + 1.1E(X_2) + 1.8E(X_3) - 13 = 2.1$$

and

$$V(Y) = V(X_1) + V(X_2) + V(X_3) + 9V(X_4) = 64.63.$$  

Therefore, we have

$$P(|1.4X_1 + 1.1X_2 + 1.9X_3 - 10| \leq 5) = P(|Y| \leq 5) = P(-5 \leq Y \leq 5) = P\left(\frac{5 - 2.1}{\sqrt{23.95}}\right) - \left(\frac{-5 - 2.1}{\sqrt{23.95}}\right) = \Phi(0.36) - \Phi(-0.88) = 0.4511.$$  

(c) (2 points). Compute $P(X_1 + 2X_2 \leq 3X_4 + 12)$.

Let $Y = X_1 + 2X_3 - 3X_4$. Then $E(Y) = 3$ and $V(Y) = 90$. Thus,

$$P(X_1 + 2X_2 \leq 3X_4 + 12) = P(Y \leq 12) = \Phi\left(\frac{12 - 3}{\sqrt{90}}\right) = \Phi(0.5) = 0.8289.$$  

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