There are totally 37 points in the exam. The students with score higher than or equal to 35 points will receive 35 points. Please write down your name, student ID number below.

NAME: ________________________________
ID: __________________________

Section 11: __ 8:30-9:20MWF
Section 3: __ 9:30-10:20MWF

This is the summary of the midterm
Min. 1st Qu. Median Mean 3rd Qu. Max.
8.50 27.00 32.50 29.84 35.00 35.00

This is the distribution of the midterm
8.5 9.0 10.5 11.0 13.0 13.0 14.5 17.0 17.0 18.0 18.0 19.0 23.0 23.0 23.0
24.5 24.5 24.5 25.0 25.0 25.0 25.0 25.5 25.5 25.5 26.0 26.0 26.0 27.0 27.0 27.0
28.0 28.0 28.0 28.0 28.5 28.5 29.0 29.0 29.0 29.5 29.5 30.0 30.0 30.5 30.5
30.5 30.5 30.5 31.0 31.0 31.0 31.5 32.0 32.0 32.5 32.5 32.5 32.5 32.5 33.0 33.0
33.0 33.0 33.0 33.0 34.0 34.0 34.5 34.5 34.5 34.5 34.5 35.0 35.0 35.0 35.0
35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0
35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0 35.0
1. (17 points, 1 point each). Fill in the blanks. Your answer should be real numbers.

(a) (3 points). Let \( A \) and \( B \), and \( C \) be events. Assume that \( P(A) = 0.28 \), \( P(B) = 0.36 \), and \( P(A \cap B) = 0.19 \). Then, \( P(A') = \) ________, \( P(A \cup B) = \) ________, and \( P(A|B) = \) ________.

(b) (3 points). Let \( A \) and \( B \) be events. Assume that \( P(A) = 0.55 \), \( P(B|A) = 0.36 \), and \( P(B|\bar{A}) = 0.84 \). Then, \( P(A \cap B) = \) ________, \( P(B) = \) ________, and \( P(A|B) = \) ________.

(c) (3 points). Let \( A \), \( B \), and \( C \) be independent events with \( P(A) = 0.3 \), \( P(B) = 0.6 \), and \( P(C) = 0.75 \). Then, \( P(A' \cap B \cap C') = \) ________, \( P(A \cup B \cup C) = \) ________, and \( P(A|B \cup C) = \) ________.

(d) (4 points). Flip a balanced die 5 times. Let \( X \) be the total number of 5 or 6. Then, \( X \sim \) ________, \( P(X \leq 1) = \) ________, \( E(X) = \) ________, and \( V(X) = \) ________.

(e) (4 points). Suppose \( X \sim \text{N}(3.7, 1.28) \). Let \( Y = -0.5X + 1.4 \). Then, \( P(X \leq 5.0) = \) ________, \( E(Y) = \) ________, \( V(Y) = \) ________, and \( Y \sim \) ________.

Answer: (a) 0.72, 0.45, 0.5278. (b) 0.198, 0.576, 0.3438. (c) 0.105, 0.93, 0.3. (d) \( \text{Bin}(5, 1/3) \), 0.4609, 1.667, and 1.111. (e) 0.8749, -0.45, 0.32, and \( \text{N}(-0.45, 0.32) \).
2. (8 points). Suppose the joint PMF of $X$ and $Y$ is

<table>
<thead>
<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
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<th>3</th>
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</thead>
<tbody>
<tr>
<td>0</td>
<td>0.04</td>
<td>0.09</td>
<td>0.10</td>
<td>0.12</td>
</tr>
<tr>
<td>1</td>
<td>0.07</td>
<td>0.11</td>
<td>0.08</td>
<td>0.04</td>
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<tr>
<td>2</td>
<td>0.15</td>
<td>0.10</td>
<td>0.05</td>
<td>0.05</td>
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(a) (2 points). Compute the marginal PMFs of $X$ and $Y$, respectively.

*Solution:* The marginal PMF of $X$ is

<table>
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<tr>
<th>$x$</th>
<th>0</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_X(x)$</td>
<td>0.35</td>
<td>0.30</td>
<td>0.35</td>
</tr>
</tbody>
</table>

and the marginal PMF of $Y$ is

<table>
<thead>
<tr>
<th>$y$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p_Y(y)$</td>
<td>0.26</td>
<td>0.30</td>
<td>0.23</td>
<td>0.21</td>
</tr>
</tbody>
</table>

(b) (2 points). Compute $E(X)$, $E(Y)$, $V(X)$, and $V(Y)$.

*Solution:* By the marginal PMF of $X$, we obtain $E(X) = 1$ and $V(X) = 0.7$. By the marginal PMF of $Y$, we obtain $E(Y) = 1.39$ and $V(Y) = 1.1779$.

(c) (2 points). Compute $\text{Cov}(X, Y)$ and $\text{Corr}(X, Y)$.

*Solution:* By the marginal PMFs, we obtain

$$E(XY) = 0.11 + 2(0.08) + 3(0.04) + 2(0.10) + 4(0.05) + 6(0.05) = 1.09.$$ 

Thus,

$$\text{Cov}(X, Y) = E(XY) - 1 \times 1.39 = -0.3$$

and

$$\text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{V(X)V(Y)}} = -0.3 \sqrt{0.7 \times 1.1779} = -0.3304.$$ 

(d) (2 points). Are $X$ and $Y$ independent. Justify your answer.

*Solution:* Since $\text{Cov}(X, Y) \neq 0$, we conclude that $X$ and $Y$ are not independent.
3. (6 points). Use the Central Limit Theorem (CLT) to compute the following probabilities. You do not need to consider 0.5-shift.

(a) (2 points). Flip a balanced coin 5000 times. Let \( X \) be the total number of heads. Compute \( P(2400 \leq X \leq 2600) \).

Solution: By \( X \sim N(5000, 1250) \), we obtain \( E(X) = 2500 \) and \( V(X) = 1250 \). Thus,

\[
P(2400 \leq X \leq 2600) \approx P(2400 \leq \mu + \sigma z \leq 2600)
\]

\[
= \Phi\left(\frac{2600 - 2500}{\sqrt{1250}}\right) - \Phi\left(\frac{2400 - 2500}{\sqrt{1250}}\right)
\]

\[
= \Phi(2.82) - \Phi(-2.82)
\]

\[
= 0.9976 - 0.0024
\]

\[
= 0.9952.
\]

(b) (2 points). Let \( X_1, \ldots, X_n \) be identically and independently observed with the common expected value \( \mu = E(X_i) = 0.7 \) and the common variance \( \sigma^2 = V(X_i) = 1.9 \). Compute \( P(0.69 \leq \bar{X} \leq 0.71) \) for \( n = 10^3, 10^4, \) and \( 10^5 \), respectively.

Solution: By the CLT, we obtain \( \bar{X} \sim \text{approx} N(0.7, 1.9/\sqrt{n}) \). Thus,

\[
P(0.69 \leq \bar{X} \leq 0.71) \approx P(0.69 \leq \mu + \sigma z \leq 0.71)
\]

\[
= \Phi\left(\frac{0.71 - 0.7}{\sqrt{1.9/\sqrt{n}}}\right) - \Phi\left(\frac{0.69 - 0.7}{\sqrt{1.9/\sqrt{n}}}\right)
\]

\[
= \begin{cases} 
0.1815, & \text{when } n = 10^3, \\
0.5318, & \text{when } n = 10^4, \\
0.9782, & \text{when } n = 10^5.
\end{cases}
\]

(c) (2 points). Let \( X_1, \ldots, X_{1000} \) be identically and independently observed with the common PMF given by

<table>
<thead>
<tr>
<th>( x )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>( p(x) )</td>
<td>0.25</td>
<td>0.15</td>
<td>0.2</td>
<td>0.4</td>
</tr>
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Compute \( P(2.70 \leq \bar{X} \leq 2.80) \).

Solution: By the PMF, we have \( \mu = 2.75 \) and \( \sigma^2 = 1.4875 \). Thus,

\( \bar{X} \sim \text{approx} N(2.75, \frac{1.4875}{1000}) = N(2.75, 0.0014875) \).

Then, we obtain

\[
P(2.70 \leq \bar{X} \leq 2.80) \approx P(2.70 \leq \mu + \sigma z \leq 2.80)
\]

\[
= \Phi\left(\frac{2.75 - 2.7}{0.03857}\right) - \Phi\left(\frac{2.65 - 2.7}{0.03857}\right)
\]

\[
= \Phi(1.30) - \Phi(-1.30)
\]

\[
= 0.8064.
\]
4. (6 points). Let \( X_1 \sim N(0.7, 1.4) \), \( X_2 \sim N(-0.5, 1.2) \), \( X_3 \sim N(0.9, 1.7) \), and \( X_4 \sim N(-0.2, 2.1) \), independently.

(a) (2 points). Compute \( P(-0.5X_1 - X_2 + 1.5X_3 - 1.2X_4 \leq 5) \).

Solution: Let \( Y = -0.5X_1 - X_2 + 1.5X_3 - 1.2X_4 \). Then, \( E(Y) = 1.74 \), \( V(Y) = 8.399 \), and \( Y \sim N(1.74, 8.399) \). Thus

\[
P(-0.5X_1 - X_2 + 1.5X_3 - 1.2X_4 \leq 5) = P(N(1.74, 8.399) \leq 5) = \Phi\left(\frac{5 - 1.74}{\sqrt{8.399}}\right) = \Phi(1.12) = 0.8686.
\]

(b) (2 points). Compute \( P(|2.5X_1 + 0.5X_2 - 2.2X_3 - 1.8X_4 - 2| \leq 4) \).

Solution: Let \( Y = 2.5X_1 + 0.5X_2 - 2.2X_3 - 1.8X_4 - 2 \). Then, \( E(Y) = -2.21 \) and \( V(Y) = 24.08 \). Thus, \( Y \sim N(-2.21, 24.08) \) and

\[
P(|2.5X_1 + 0.5X_2 - 2.2X_3 - 1.8X_4 - 2| \leq 4) = P(-4 \leq N(-2.21, 24.08) \leq 4) = \Phi\left(\frac{-4 + 2.21}{\sqrt{24.08}}\right) - \Phi\left(\frac{-4 + 2.21}{\sqrt{24.08}}\right) = \Phi(1.27) - \Phi(-0.36) = 0.5385.
\]

(c) (2 points). Compute \( P(2X_1 - 1.2X_2 \leq 3.1X_3 - 1.4X_4 + 9) \).

Solution: Let \( Y = 2X_1 - 1.2X_2 - 3.1X_3 + 1.4X_4 \). Then, \( E(Y) = -1.07 \) and \( V(Y) = 27.78 \). Thus, \( Y \sim N(-1.07, 27.78) \) and

\[
P(2X_1 - 1.2X_2 \leq 3.1X_3 - 1.4X_4 + 3) = P(Y \leq 9) = P(N(-1.07, 27.78) \leq 9) = \Phi\left(\frac{9 + 1.07}{\sqrt{27.78}}\right) = \Phi(1.91) = 0.9719.
\]