Section 8.2: Test about a Population Mean
Case 1: A Normal Population with known $\sigma$.

Assume data $X_1, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ and $\sigma$ is known. Note that if $\mu = \mu_0$, then

$$Z = \frac{\bar{X} - \mu_0}{\sigma / \sqrt{n}} \sim N(0, 1).$$

The test statistic is

$$z = \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}}.$$

Let $\alpha$ be the significance level. Then,

(a) for $H_0 : \mu = \mu_0$ (or $H_0 : \mu \leq \mu_0$) versus $H_a : \mu > \mu_0$, we reject $H_0$ if $z > z_\alpha$;

(b) for $H_0 : \mu = \mu_0$ (or $H_0 : \mu \geq \mu_0$) versus $H_a : \mu < \mu_0$, we reject $H_0$ if $z < -z_\alpha$;

(c) for $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$, we reject $H_0$ if $|z| > z_{\alpha/2}$. 


Sample Size Determination. The type II error probability is

\[
\begin{cases}
\Phi(z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}), & \text{when } \mu' > \mu_0 \text{ in (a)} \\
1 - \Phi(-z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}), & \text{when } \mu' < \mu_0 \text{ in (b)}
\end{cases}
\]

for one-tailed test, and is

\[
\Phi(z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}) - \Phi(-z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}),
\]

when \( \mu' \neq \mu_0 \) in (c) for a two-sided test. Then, if we require type II error probability less than \( \beta \), we need at least

\[
n = \begin{cases}
\left[ \frac{\sigma(z_\alpha + z_\beta)}{\mu_0 - \mu'} \right]^2 & \text{for (a) or (b)} \\
\left[ \frac{\sigma(z_{\alpha/2} + z_\beta)}{\mu_0 - \mu'} \right]^2 & \text{for (c)}
\end{cases}
\]
Case 2: Large Sample Test.

Assume data $X_1, \ldots, X_n$ are iid with mean $\mu$ and variance $\sigma^2$. Note when $n$ is large (e.g. $n > 40$), we have

$$z = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} \sim \text{approx} \ N(0, 1).$$

Let $\alpha$ be the significance level. Then,

(a) for $H_0 : \mu = \mu_0$ (or $H_0 : \mu \leq \mu_0$) versus $H_a : \mu > \mu_0$, we reject $H_0$ if $z > z_\alpha$;

(b) for $H_0 : \mu = \mu_0$ (or $H_0 : \mu \geq \mu_0$) versus $H_a : \mu < \mu_0$, we reject $H_0$ if $z < -z_\alpha$;

(c) for $H_0 : \mu = \mu_0$ versus $H_a : \mu \neq \mu_0$, we reject $H_0$ if $|z| > z_{\alpha/2}$. 
First example of Section 8.2: example 8.6 on textbook. Statement: the true average system-activation temperature is $130^\circ F$ ($\mu = 130$). A sample of $n = 9$ report $\bar{x} = 131.08^\circ F$. Assume $\sigma = 1.5^\circ F$. Q: does the evidence contradict the claim at significance level $\alpha = 0.01$. Assume normal.

- We need to test $H_0: \mu = 130$ versus $H_a: \mu \neq 130$. Thus, $\mu_0 = 130$.

- Test statistic

$$z = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}} = \frac{131.08 - 130}{1.5/\sqrt{9}} = 2.16$$

- Since 2.16 is not $> z_{0.005} = 2.58$ in absolute value, we claim the evidence does not contradict the claim.
Second example of Section 8.2: example 8.7 on textbook. Assume normal. Consider the test for

\[ H_0 : \mu \leq 30000 \leftrightarrow H_a : \mu > 30000. \]

Assume \( n = 16 \) and \( \sigma = 1500 \). Then, \( \mu_0 = 30000 \) and this is testing problem (a). When \( n = 16 \), for \( \alpha = 0.01 \) we reject \( H_0 \) if

\[
\frac{\bar{x} - 30000}{1500/\sqrt{16}} = \frac{\bar{x} - 30000}{375} > z_{0.01} = 2.33
\]

which is equivalent to that we reject \( H_0 \) if

\[ \bar{x} > 30000 + 2.33 \times 375 = 30873.75. \]

If we want

\[ \beta(31000) = 0.1, \]

we need

\[
n \geq \left[ \frac{1500(z_{0.01} + z_{0.1})}{30000 - 31000} \right]^2 = 29.32
\]

and thus we choose \( n = 30 \).
Third example of Section 8.2: example 8.8 on textbook. Without assuming normal. A sample of $n = 52$ reports $\bar{x} = 28.76$ and $s = 12.2647$. Suppose we test

$$H_0 : \mu \geq 30 \leftrightarrow H_a : \mu < 30.$$  

Since

$$z = \frac{\bar{x} - 30}{s/\sqrt{n}} = \frac{28.76 - 30}{12.2647/\sqrt{52}} = -0.73 \not< -1.645$$

we fail to reject $H_0$ at $\alpha = 0.05$. Thus, we conclude $\mu = 30$.  