Section 4.3: The Normal Distribution
Section 4.3 has the following concepts:

- A probability density function \( PDF \) of a continuous random variable \( X \) has the property
  \[
P(X \leq a) = F(a) = \int_{-\infty}^{a} f(x) \, dx,
\]
  where \( F(x) \) is the CDF of \( X \).

- If for \(-\infty < \mu < \infty \) and \( \sigma > 0 \)
  \[
f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}},
\]
  we call \( X \) following a normal distribution with expected value \( \mu \) and variance \( \sigma^2 \), and denote
  \[X \sim N(\mu, \sigma^2)\].

- If \( \mu = 0 \) and \( \sigma^2 = 1 \), then we call \( X \) following a standard normal distribution. In this case, we have
  \[X \sim N(0, 1)\].

- The density of \( N(0, 1) \) is
  \[
f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}.
\]

- Let \( F(x) \) be the CDF of a continuous random variable \( X \), then \( F^{-1} \) is called the quantile (or percentile) function of \( X \).
Section 4.3 has the following formulae:

- If $X \sim N(\mu, \sigma^2)$, then
  \[ Z = \frac{X - \mu}{\sigma} \sim N(0, 1). \]

- If $X \sim N(0, 1)$, then we write $\Phi$ and $\phi$ and the CDF and PDF of $X$, which means
  \[ \Phi(z) = P(N(0, 1) \leq z). \]

- $\Phi$ is symmetric about 0, i.e.,
  \[ \Phi(-z) = 1 - \Phi(z). \]

- We can use Table A.3 on page 740(v6)/668(v7) to find the values of $\Phi(z)$ for a given $z$ and $\Phi^{-1}(q)$ for a given $q$.

- We denote $z_\alpha$ as the solution to
  \[ \Phi(z_\alpha) = 1 - \alpha. \]

- In addition, this table can also be used for general $N(\mu, \sigma^2)$ by
  \[ P(N(\mu, \sigma^2) \leq a) = \Phi\left(\frac{a - \mu}{\sigma}\right). \]
• The 1 − \( \alpha \) quantile (percentile) of \( N(\mu, \sigma^2) \) is
  \[ \mu + \sigma z_\alpha. \]

  This means
  \[ P(N(\mu, \sigma^2) \leq \mu + \sigma z_\alpha) = 1 - \alpha. \]

• If \( X \sim Bin(n, p) \), then when \( n \) is large
  \[ Z = \frac{X - np}{\sqrt{np(1-p)}} \sim \text{approx} \ N(0, 1). \]

• Since binomial is discrete and normal is continuous, we recommend a 0.5 shift in the computation. That is if \( X \sim Bin(n, p) \), then we can approximate
  \[ P(X \leq a) \approx \Phi\left( \frac{a + 0.5 - np}{\sqrt{np(1-p)}} \right) \]
  for \( a = 0, 1, 2 \ldots, n. \)

• Here, 0.5 shift is also called the continuity correction.
Standard normal is symmetric about 0.

Thus, we have

\[ P(N(0, 1) \geq a) = P(N(0, 1) \leq -a). \]
Graph of General Normal Density \((N(4, 2^2))\)

- General normal is symmetric about \(\mu\).
- Thus, we have

\[
P(N(\mu, \sigma^2) \geq \mu + a) = P(N(\mu, \sigma^2) \leq \mu - a).
\]
First example of Section 4.3: example 4.13 on textbook. Use table A.3 to find the following probabilities. Assume $Z \sim N(0, 1)$.

1. 

$$P(Z \leq 1.25) = 0.8944.$$ 

2. 

$$P(Z > 1.25) = 1 - \Phi(1.25) = 1 - 0.8944 = 0.1056.$$ 

3. Since $\Phi$ is symmetric

$$P(Z \leq -1.25) = \Phi(Z \geq 1.25) = 0.1056.$$ 

4. For normal distribution, end points included or not is not an issue here.

$$P(-0.38 \leq Z \leq 1.25) = P(-0.38 \leq Z < 1.25) = P(-0.38 < Z \leq 1.25) = P(-0.38 < Z < 1.25) = \Phi(1.25) - \Phi(-0.38) = 0.8944 - 0.3520 = 0.5424.$$
Second example of Section 4.3: example 4.14 on textbook.

- Find a $z$ so that $\Phi(z) = 0.99$.

  Answer: since in Table A.3

  \[
  \Phi(2.33) = 0.9901
  \]

  and

  \[
  \Phi(2.32) = 0.9898
  \]

  we have

  \[
  2.32 < z < 2.33.
  \]

  Roughly, we can choose

  \[
  z = 2.33.
  \]

  Then,

  \[
  z_{0.01} = 2.33.
  \]
Third example of Section 4.3: example 4.15 on textbook. This table is useful:

<table>
<thead>
<tr>
<th>$q$</th>
<th>0.9</th>
<th>0.95</th>
<th>0.975</th>
<th>0.99</th>
<th>0.995</th>
<th>0.999</th>
<th>0.9995</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\alpha$</td>
<td>0.1</td>
<td>0.05</td>
<td>0.025</td>
<td>0.01</td>
<td>0.005</td>
<td>0.001</td>
<td>0.0005</td>
</tr>
<tr>
<td>$z_{\alpha}$</td>
<td>1.28</td>
<td>1.645</td>
<td>1.96</td>
<td>2.33</td>
<td>2.58</td>
<td>3.08</td>
<td>3.27</td>
</tr>
</tbody>
</table>

This table tells us that $\Phi(1.28) = 0.9$; $\Phi(1.645) = 0.95$, $\Phi(1.96) = 0.975$ and so on.
Fourth example of Section 4.3: example 4.16 on textbook.

- It is known the mean value $\mu = 1.25$, the standard deviation is 0.46, and assume normal.
- Then, $\mu = 1.25$ and $\sigma^2 = 0.46^2$. Thus,

$$X \sim N(1.25, 0.46^2).$$

- Therefore, we have

$$P(1 \leq X \leq 1.75) = P(X \leq 1.75) - P(X \leq 1)$$

$$= \Phi\left(\frac{1.75 - 1.25}{0.46}\right) - \Phi\left(\frac{1 - 1.25}{0.46}\right)$$

$$= \Phi(1.09) - \Phi(-0.54)$$

$$= 0.8621 - 0.2946$$

$$= 0.5675.$$

- Note, it does not matter points 1 and 1.75 are included or not since normal is continuous.
Fifth example of Section 4.3: example 4.17 on textbook.

- Assume $X \sim N(\mu, \sigma^2)$,

  $P(X \text{ is within one std dev of its mean})$

  $= P(\mu - \sigma \leq X \leq \mu + \sigma)$

  $= P(X \leq \mu + \sigma) - P(X \leq \mu - \sigma)$

  $= \Phi(1) - \Phi(-1)$

  $= 0.6826$.

- Similarly, we have

  $P(\mu - 2\sigma \leq X \leq \mu + 2\sigma) = 0.9545$

  $P(\mu - 3\sigma \leq X \leq \mu + 3\sigma) = 0.9973$

  and

  $P(\mu - 4\sigma \leq X \leq \mu + 4\sigma) = 0.9999$. 
Sixth example of Section 4.3: example 4.18 on textbook. Suppose $\mu = 64$ and $\sigma = 0.78$. Then, we have

$$X \sim N(64, 0.78^2).$$

Find a $c$ so that $P(X > c) = 0.005$.

Answer: Since

$$P(X > c) = 1 - P(X \leq c) = 1 - \Phi\left(\frac{c - 64}{0.78}\right)$$

we have

$$\Phi\left(\frac{c - 64}{0.78}\right) = 0.995$$

$$\Rightarrow \frac{c - 64}{0.78} = 2.58$$

$$\Rightarrow c = 64 + 2.58 \times 0.78 = 66.01.$$
Seventh example of Section 4.3: example 4.20 on textbook. In this problem, we have

\[ X \sim Bin(50, 0.25) \]

Then,

\[ E(X) = 50 \times 0.25 = 12.5 \]

and

\[ V(X) = 50 \times 0.25 \times 0.75 = 9.375. \]

Therefore,

\[ X \sim^{approx} N(12.5, 9.375). \]

Adjusting the probability by 0.5 shift (continuity correction), we have

\[
P(X \leq 10) \approx \Phi\left(\frac{10 + 0.5 - 12.5}{\sqrt{9.375}}\right)
= \Phi(-0.65) = 0.2578;
\]

\[
P(X < 10) \approx \Phi\left(\frac{10 - 0.5 - 12.5}{\sqrt{9.375}}\right)
= \Phi(-0.98) = 0.1635;
\]
\[ P(5 \leq X \leq 15) \]
\[ = P(X \leq 15) - P(X \leq 4) \]
\[ \approx \Phi\left(\frac{15 + 0.5 - 12.5}{\sqrt{9.375}}\right) - \Phi\left(\frac{4 + 0.5 - 12.5}{\sqrt{9.375}}\right) \]
\[ = \Phi(0.98) - \Phi(-2.61) \]
\[ = 0.8320. \]

\[ P(5 \leq X < 15) \]
\[ = P(X \leq 14) - P(X \leq 4) \]
\[ \approx \Phi\left(\frac{14 + 0.5 - 12.5}{\sqrt{9.375}}\right) - \Phi\left(\frac{4 + 0.5 - 12.5}{\sqrt{9.375}}\right) \]
\[ = \Phi(0.65) - \Phi(-2.61) \]
\[ = 0.7376. \]
\[ P(5 < X \leq 15) \]
\[ = P(X \leq 15) - P(X \leq 5) \]
\[ \approx \Phi\left(\frac{15 + 0.5 - 12.5}{\sqrt{9.375}}\right) - \Phi\left(\frac{5 + 0.5 - 12.5}{\sqrt{9.375}}\right) \]
\[ = \Phi(0.98) - \Phi(-2.29) \]
\[ = 0.8254. \]

\[ P(5 < X < 15) \]
\[ = P(X \leq 14) - P(X \leq 5) \]
\[ \approx \Phi\left(\frac{14 + 0.5 - 12.5}{\sqrt{9.375}}\right) - \Phi\left(\frac{5 + 0.5 - 12.5}{\sqrt{9.375}}\right) \]
\[ = \Phi(0.65) - \Phi(-2.29) \]
\[ = 0.7311. \]