Section 3.3: Expected value of Discrete Random Variables
Section 3.3 introduces the following concepts and formulae:

- Suppose the PMF of $X$ is given by $p(a) = P(X = a)$.
- Expected value (expectation or mean): defined by
  \[ \mu = E(X) = \sum ap(a). \]
- Expected value of a function of random variable:
  \[ E[h(X)] = \sum h(a)p(a) \]
  for any function $h$.
- Rules of expected value: for two constants $k$ and $b$, there is
  \[ E(kX + b) = kE(X) + b = k\mu + b. \]
- Variance.
  \[ V(X) = E[(X - \mu)^2]. \]
- Rules of variance: for any two constants $k$ and $b$, there is
  \[ V(kX + b) = k^2V(X). \]
First example of Section 3.3:

- The PMF of $X$ is

\[
\begin{array}{cccccccc}
a & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
p(a) & 0.01 & 0.03 & 0.13 & 0.25 & 0.39 & 0.17 & 0.02 \\
\text{Count} & 150 & 450 & 1950 & 3750 & 5850 & 2550 & 300 \\
\end{array}
\]

Total count is 15000.

- Expected value of $X$ is the weighted average, it can be

\[
\mu = 1(0.01) + 2(0.03) + \cdots + 7(0.02) = 4.57
\]

or

\[
\mu = 1\left(\frac{150}{15000}\right) + 2\left(\frac{450}{15000}\right) + \cdots + 7\left(\frac{300}{15000}\right) = 4.57.
\]
Second example of Section 3.3: example 3.17.

- The PMF of \( X \) is given by

<table>
<thead>
<tr>
<th>a</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(a)</td>
<td>0.002</td>
<td>0.001</td>
<td>0.002</td>
<td>0.005</td>
<td>0.02</td>
<td>0.04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>a</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>p(a)</td>
<td>0.18</td>
<td>0.37</td>
<td>0.25</td>
<td>0.12</td>
<td>0.01</td>
</tr>
</tbody>
</table>

- The expected value is

\[
E(X) = \mu = 0(0.002) + 1(0.001) + 2(0.002) + \cdots + 8(0.25) + 9(0.12) + 10(0.01)
\]

\[= 7.15.\]

- \( E(X^2) \)

\[
= 0^2(0.002) + 1^2(0.001) + 2^2(0.002) + \cdots + 8^2(0.25) + 9^2(0.12) + 10^2(0.01)
\]

\[= 52.704.\]

- The variance is

\[
V(X) = \sigma^2 = 52.704 - 7.15^2 = 1.5815
\]

- The standard deviation is

\[\sigma = \sqrt{1.5815} = 1.2576.\]
Third example of Section 3.3: example 3.18.

- The PMF of $X$ is
  \[
  p(x) = \begin{cases} 
  1 - p, & x = 0 \\
  p, & x = 1 \\
  0, & \text{otherwise}
  \end{cases}
  \]

- Then,
  \[
  E(X) = 0(1 - p) + 1(p) = p
  \]
  and
  \[
  E(X^2) = 0^2(1 - p) + 1^2(p) = p.
  \]
  Thus,
  \[
  V(X) = p - p^2 = p(1 - p).
  \]
Fourth example of Section 3.3: example 3.20.

- The PMF of $X$ is
  $$p(x) = \begin{cases} \frac{k}{x^2} & x = 1, 2, 3, \cdots \\ 0, & \text{otherwise} \end{cases}$$

- Then, we have
  $$\sum_{x=1}^{\infty} p(x) = 1$$
  $$\Rightarrow k = \left[ \sum_{x=1}^{\infty} \frac{1}{x^2} \right]^{-1} = \frac{6}{\pi^2}.$$

- Since
  $$\sum_{x=1}^{\infty} x p(x) = \frac{6}{\pi^2} \sum_{x=1}^{\infty} \frac{1}{x} = \infty$$
  the expected value $E(X)$ does not exist.
Fifth example of Section 3.3: example 3.22.

- The PMF of $X$ is

<table>
<thead>
<tr>
<th>$x$</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(x)$</td>
<td>0.5</td>
<td>0.3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

- Then,

$$E(X) = 4(0.5) + 6(0.3) + 8(0.2) = 5.4.$$ 

- Suppose $Y = h(X) = 20 + 3X + 0.5X^2$. Then, the PMF of $Y$ is

<table>
<thead>
<tr>
<th>$y$</th>
<th>40</th>
<th>56</th>
<th>76</th>
</tr>
</thead>
<tbody>
<tr>
<td>$p(y)$</td>
<td>0.5</td>
<td>0.3</td>
<td>0.2</td>
</tr>
</tbody>
</table>

- Thus,

$$E(Y) = E[h(Y)]$$

$$= (40)(0.5) + (56)(0.3) + (76)(0.2)$$

$$= h(4)0.5 + h(6)0.3 + h(8)0.2$$

$$= 52.$$
Sixth example of Section 3.3: example 3.10 again.

- The PMF of $X$ is

\[
\begin{array}{cccc}
  x & 1 & 2 & 3 & 4 \\
p(x) & 0.1 & 0.2 & 0.3 & 0.4 \\
\end{array}
\]

- Then, $E(X) = 3$ and $V(X) = 1$.

- Let $Y = h(X) = 800X - 900$. Then,

\[
E(Y) = 800E(X) - 900 = 1500
\]

and

\[
V(Y) = 800^2V(X) = 800^2.
\]
Seventh example of Section 3.3: Flip a dice twice and let $X$ be the sum of the two outcomes. Then, $X$ is a discrete random variable.

- Compute the probability mass function (PMF) of $X$.

  Answer:

  $\begin{array}{ccccccc}
  x & 2 & 3 & 4 & 5 & 6 & 7 \\
  p(x) & \frac{1}{36} & \frac{2}{36} & \frac{3}{36} & \frac{4}{36} & \frac{5}{36} & \frac{6}{36} \\
  \hline
  x & 8 & 9 & 10 & 11 & 12 \\
  p(x) & \frac{5}{36} & \frac{4}{36} & \frac{3}{36} & \frac{2}{36} & \frac{1}{36} \\
  \end{array}$

- Use PMF to compute $P(2 \leq X \leq 4)$, $P(2 \leq X < 4)$, $P(X \geq 4)$. Answer:

  \[
  P(2 \leq X \leq 4) = p(2) + p(3) + p(4) = \frac{6}{36}
  \]

  \[
  P(2 \leq X < 4) = P(2) + P(3) = \frac{3}{36}
  \]

  and

  \[
  P(X \geq 4) = p(4) + p(5) + \cdots + p(12) = \frac{33}{36}.
  \]
• Compute the cumulative distribution function (CDF) of $X$.

Answer

$$F(x) = \begin{cases} 
0 & x < 2 \\
1/36 & 2 \leq x < 3 \\
3/36 & 3 \leq x < 4 \\
6/36 & 4 \leq x < 5 \\
10/36 & 5 \leq x < 6 \\
15/36 & 6 \leq x < 7 \\
21/36 & 7 \leq x < 8 \\
26/36 & 8 \leq x < 9 \\
30/36 & 9 \leq x < 10 \\
33/36 & 10 \leq x < 11 \\
35/36 & 11 \leq x < 12 \\
1 & x \geq 12 
\end{cases}$$

• Use CDF to compute $P(2 \leq X \leq 4)$, $P(2 \leq X < 4)$, $P(X \geq 4)$.

Answer:

$$P(2 \leq X \leq 4) = F(4) - F(1) = \frac{6}{36}$$

$$P(2 \leq X < 4) = F(3) - F(1) = \frac{3}{36}$$

and

$$P(X \geq 4) = 1 - F(3) = \frac{33}{36}.$$
• Compute $E(X)$.

$$E(X) = 2\left(\frac{1}{36}\right) + 3\left(\frac{2}{36}\right) + \cdots + 12\left(\frac{1}{36}\right) = 7.$$  

• Compute $V(X)$.

$$E(X^2) = 2^2\left(\frac{1}{36}\right)+3^2\left(\frac{2}{36}\right)+\cdots+12^2\left(\frac{1}{36}\right) = 54.83.$$  

Then

$$V(X) = 54.83 - 7^2 = 5.83.$$  

• Compute the standard deviation of $X$.

$$\sqrt{V(X)} = \sqrt{5.83} = 2.415.$$
Eighth example of Section 3.3: Suppose the CDF of a random variable $X$ is given by

$$F(x) = \begin{cases} 
0, & \text{when } x < 0 \\
0.2, & \text{when } 0 \leq x < 1 \\
0.35, & \text{when } 1 \leq x < 2, \\
0.65, & \text{when } 2 \leq x < 3 \\
0.85 & \text{when } 3 \leq x < 4 \\
1, & \text{when } x \geq 4.
\end{cases}$$

- Compute the PMF of $X$. Answer:

$$p(0) = F(0) - F(-1) = 0.2 - 0 = 0.2$$

$$p(1) = F(1) - F(0) = 0.35 - 0.2 = 0.15$$

$$p(2) = F(2) - F(1) = 0.65 - 0.35 = 0.3$$

$$p(3) = F(3) - F(2) = 0.85 - 0.65 = 0.2$$

and

$$p(4) = F(4) - F(3) = 1 - 0.85 = 0.15.$$
• Compute $P(1 \leq X \leq 3)$, $P(1 < X < 3)$, $P(1 < X \leq 3)$, $P(1 \leq X < 3)$.

Answer:

$P(1 \leq X \leq 3) = F(3) - F(0) = 0.85 - 0.2 = 0.65$

$P(1 < X < 3) = F(2) - F(1) = 0.65 - 0.35 = 0.3$

$P(1 < X \leq 3) = F(3) - F(1) = 0.85 - 0.35 = 0.5.$

$P(1 \leq X < 3) = F(2) - F(0) = 0.65 - 0.2 = 0.45.$

• Compute $E(X)$.

Answer: $E(X) = 1.95$

• Compute $V(X)$.

Answer: $E(X^2) = 5.55$, $V(X) = 1.7475$

• Compute the standard deviation of $X$.

Answer: $\sqrt{1.7475} = 1.3219.$