Section 2.5:

Independence
• Definition: For any two events $A$ and $B$ with $P(B) > 0$, we say $A$ and $B$ are independent if

$$P(A|B) = P(A).$$

• Proposition:

$$P(A|B) = P(A) \iff P(A \cap B) = P(A)P(B) \iff P(B|A) = P(B).$$

• Proposition: if $A$ and $B$ are independent, then

$$P(A \cap B) = P(A)P(B)$$

$$P(A \cap B') = P(A)P(B')$$

$$P(A' \cap B) = P(A')P(B)$$

$$P(A' \cap B') = P(A')P(B').$$

• Definition: we say $A_1, \ldots, A_n$ are mutually independent if for every $k$, and every subset of indices $i_1, i_2, \ldots, i_k$, there is

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1})P(A_{i_2}) \cdots P(A_{i_k}).$$
First example of Section 2.5.

Review of “without replacement” and “with replacement”. A bag has 10 blue balls and 8 red balls. Randomly pick up three of them.

Let $A_k$ be the $k$-th is red.

(a) What is the probability that the first one is red.

\[ P(A_1) = \frac{8}{18} = \frac{4}{9}. \]
(b) Compute the probability that given the first is red, the second one is also red for without replacement or with replacement respectively

Without replacement:

\[ P(A_1 \cap A_2) = \frac{\binom{8}{2}}{\binom{18}{2}} = \frac{28}{153}, \]

and

\[ P(A_2|A_1) = \frac{(28/153)}{(4/9)} = \frac{7}{17}. \]

With replacement:

\[ P(A_1 \cap A_2) = \frac{8^2}{18^2} = \frac{16}{81}, \]

and

\[ P(A_2|A_1) = \frac{(16/81)}{(4/9)} = \frac{4}{9}. \]
(c) Compute the probability that given the first is blue, the second one is red for without replacement or with replacement respectively.

Without replacement:

\[ P(A'_1 \cap A_2) = \frac{10 \times 8}{18 \times 17} = \frac{40}{153} \]

and

\[ P(A_2|A'_1) = \frac{(40/153)}{(5/9)} = \frac{8}{17}. \]

With replacement

\[ P(A'_1 \cap A_2) = \frac{10 \times 8}{18^2} = \frac{20}{81} \]

and

\[ P(A_2|A'_1) = \frac{(20/91)}{(5/9)} = \frac{4}{9}. \]

(d) Compute the probability that the second is red for without or with replacement respectively.

Without replacement:


With replacement:

Thus, $A_1$ and $A_2$ are independent under with replacement but not independent under without replacement.
Second example of section 2.5: example 2:34 on textbook.

- Let $A$ be the event that the washer needs service while under warranty and $B$ be defined analogously for the dryer.

- $P(A) = 0.3$ and $P(B) = 0.1$.

- Assume $A$ and $B$ are independent. Then, we have

$$P(A \cap B) = P(A)P(B) = 0.03.$$  
$$P(A \cap B') = P(A)P(B') = 0.27.$$  
$$P(A' \cap B) = P(A')P(B) = 0.07.$$  
$$P(A' \cap B') = P(A')P(B') = 0.63.$$
Third example of section 2.5: example 2:36 on textbook.

There are 6 electrical components connected in the systems (see textbook). Suppose $P(A_i) = 0.9$ for $i = 1, \cdots, 6$.

- For left part, we compute

\[
P[(A_1 \cap A_2 \cap A_3) \cup (A_4 \cap A_5 \cap A_6)] = P(A_1 \cap A_2 \cap A_3) + P(A_4 \cap A_5 \cap A_6) - P(A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6)
\]

\[
= 0.9^3 + 0.9^3 - 0.9^6 = 0.9266.
\]

- For right part, we compute

\[
P[(A_1 \cup A_4) \cap (A_2 \cap A_5) \cap (A_3 \cup A_6)] = P(A_1 \cup A_4)P(A_2 \cup A_5)P(A_3 \cup A_6)
\]

\[
= [P(A_1 \cup A_4)]^3
\]

\[
= [P(A_1) + P(A_4) - P(A_1 \cap A_4)]^3
\]

\[
= (0.9 + 0.9 - 0.9^2)^3
\]

\[
= 0.9703.
\]
Fourth example of section 2.5.

Flip a die 4 times. Let $X$ be the number of 6 we see. What is the probability of $X = 0$, $X = 1$, $X = 2$, $X = 3$ and $X = 4$.

\[
P(X = 0) = \binom{4}{0} \left(\frac{1}{6}\right)^0 \left(\frac{5}{6}\right)^4 = \frac{5^4}{6^4} = \frac{625}{1296}.\]

\[
P(X = 1) = \binom{4}{1} \left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 = \frac{125}{324}.\]

\[
P(X = 2) = \binom{4}{2} \left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 = \frac{25}{216}.\]

\[
P(X = 3) = \binom{4}{3} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^1 = \frac{5}{324}.\]

\[
P(X = 4) = \binom{4}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^0 = \frac{1}{1296}.\]
Fifth example of section 2.5.

Flip a die many times until we see one six. What is the probability that the flipping time is 1, 2, 3, 4? How about the result for a general $n$?

Answer: Let $X$ be the time of flipping.

\[ P(X = 1) = \frac{1}{6} \]

\[ P(X = 2) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right) = \frac{5}{36} \]

\[ P(X = 3) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^2 = \frac{25}{216} \]

\[ P(X = 4) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3 = \frac{125}{1296} \]

and

\[ P(X = n) = \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{n-1} = \frac{5^{n-1}}{6^n}. \]