Chapter 2

Sections 2.1 and 2.2.
Section 1 introduces the following concepts.

- **Outcomes**: single point of possibility.
- **Sample Space**: the set of all outcomes (possibilities), denoted by $S$.
- **Empty set**: denoted by $\phi$.
- **Subset**: if all outcomes in $A$ are also in $B$, then $A$ is a subset of $B$ and we write $A \subseteq B$.
- **Event**: the subset of sample space, denoted by $A$. Then $A \subseteq S$.
- **Union**: for events $A$ and $B$, the union is “either $A$ or $B$” and denoted by $A \cup B$.
- **Intersection**: for events $A$ and $B$, the intersection is “both $A$ and $B$” and denoted by $A \cap B$.
- **Complement**: for event $A$, the complement is “not $A$” and denoted by $A' = \bar{A} = A^c$.
- **Mutually exclusive or disjoint**: $A$ and $B$ is “mutually exclusive” or “disjoint” if $A \cap B = \phi$. 


Section 1 has the following formulae.

- \((A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C\).

- \((A \cup B) \cup C = A \cup (B \cup C) = A \cup B \cup C\).

- \((A \cap B) \cup C = (A \cup C) \cap (B \cup C)\).

- \((A \cup B) \cap C = (A \cap C) \cup (B \cap C)\).

- \((A \cap B)' = A' \cup B'\).

- \((A \cup B)' = A' \cap B'\).
First example of Section 2.1: Let $A$, $B$ and $C$ be sets.

(a) Suppose

$$A = \{0, 1, 2, a, b, x\}$$

and

$$B = \{0, 0, 1, 2, a, b, x\}.$$ 

Is $A = B$? What are $\#A$ and $\#B$?


(b) Suppose

$$A = \{0, 1, 2, a, b, x\}$$

and

$$B = \{0, 2, 5, a\}.$$ 

Then, we have

$$A \cap B = \{0, 2, a\}$$

and

$$A \cup B = \{0, 1, 2, 5, a, b, x\}.$$
(c) Suppose

\[ A = \{1, 2, 3, a, c\}, \]

\[ B = \{0, 1, a\}, \]

and

\[ C = \{a, b, x\}. \]

Then, we have

\[ A \cup B \cup C = \{0, 1, 2, 3, a, b, c, x\} \]

and

\[ A \cap B \cap C = \{a\}. \]

(d) Suppose

\[ A = \{1, 2, 3, a, b\}, \]

\[ B = \{0, 1, 2\} \]

and

\[ C = \{a, b\}. \]

Is \( B \subseteq A \)? Is \( C \subseteq A \)?

Answer: \( B \nsubseteq A, \ C \subseteq A \).
Second example of Section 2.1

- Suppose we flip a coin twice. Denote “head” by “H” and “tail” by “T”.
- The possible outcomes are: \( HH, HT, TH, TT \).
- The sample space is
  \[
  S = \{HH, HT, TH, TT\}.
  \]
- There are 16 possible events. They could be \( \phi \), \( \{HH\} \), \( \{HH, HT\} \), \( \{HH, HT, TT\} \), and etc.
- Q: how do you understand those concepts if we flip a coin three times instead of twice?
Section 2 introduces the following concepts and formulae.

- **Probability**: there are three axioms.
  
  - $P(A) \geq 0$.
  
  - $P(S) = 1$.
  
  - If $A_1, \cdots, A_k$ is a finite collection of mutually exclusion events, then
    
    $$P(A_1 \cup A_2 \cup \cdots \cup A_k) = \sum_{i=1}^{k} P(A_i).$$
  
  - If $A_1, A_2, \cdots$ is an infinite collection of mutually exclusive events, then
    
    $$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$
• For any event \( A \),
\[
P(A) = 1 - P(A').
\]
• If \( A \) and \( B \) are mutually exclusive, then
\[
P(A \cap B) = 0.
\]
• For any two events \( A \) and \( B \),
\[
P(A \cup B) = P(A) + P(B) - P(A \cap B).
\]
• For any three events \( A \), \( B \) and \( C \),
\[
P(A \cup B \cup C) = P(A) + P(B) + P(C)
- P(A \cap B) - P(A \cap C) - P(B \cap C)
+ P(A \cap B \cap C).
\]
• If \( B \subseteq A \), then
\[
P(A \cap B') = P(A) - P(B).
\]

\[
P(A \cap B') = P(A) - P(A \cap B).
\]

• Equally likely: assume \( S \) is finite and probability of every outcome is the same. Thus,
\[
P(A) = \frac{\#(A)}{\#(S)},
\]
where \( \# \) represents the number of elements.
First example of Section 2.2: Example 2.14 on textbook.

- Two events $A$ and $B$. They are
  
  $A = \{\text{subscribes to the metropolitan paper}\}$
  and
  
  $B = \{\text{subscribes to the local paper}\}$.

- It is known that $P(A) = 0.6$, $P(B) = 0.8$ and $P(A \cap B) = 0.5$.

- The probability of subscription of at least one paper is
  
  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
  
  $= 0.6 + 0.8 - 0.5$
  
  $= 0.9$.

- The probability of subscription of exactly one is
  
  $P(\{\text{exact one}\}) = P(A \cap B') + P(A' \cap B)$
  
  $= [P(A) - P(A \cap B)] + [P(B) - P(A \cap B)]$
  
  $= (0.6 - 0.5) + (0.8 - 0.5)$
  
  $= 0.4$. 
Second example of Section 2.2: example 2.16 on textbook.

- Flip two balanced dice.
- The 36 outcomes are \((1, 1), (1, 2), \ldots, (6, 6)\).
- Let \(T\) be the total.
- Let 
  \[ A = \{T = 7\} = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\} \]
- Then, we have 
  \[ P(A) = \frac{6}{36} = \frac{1}{6}. \]
- Q: can you give a way to display the probabilities with respect to different \(T\) values?
  Answer: we may use a table like

<table>
<thead>
<tr>
<th>(T)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>(P)</td>
<td>(\frac{1}{36})</td>
<td>(\frac{2}{36})</td>
<td>(\frac{3}{36})</td>
<td>(\frac{4}{36})</td>
<td>(\frac{5}{36})</td>
<td>(\frac{6}{36})</td>
<td>(\frac{5}{36})</td>
<td>(\frac{4}{36})</td>
<td>(\frac{3}{36})</td>
<td>(\frac{2}{36})</td>
<td>(\frac{1}{36})</td>
</tr>
</tbody>
</table>

- Q: how do you understand the word “balanced”? can you propose a way to find whether a die is balanced or not?