Homework # 9, Stat355, Spring 2021

Please use R to do all of the following problems. Submit your homework by a WORD or PDF document with main results and the corresponding code.

1. Confidence intervals are often used in practice. To understand how to implement the method, compute confidence intervals in the following data.

(a) Data: $X_1, \cdots, X_n \sim iid N(\mu, \sigma^2)$. The $z$-confidence interval for $\mu$ is $\bar{x} \pm z_{\alpha/2} s/\sqrt{n}$. The $t$-confidence interval for $\mu$ is $\bar{x} \pm t_{\alpha/2, n-1} s/\sqrt{n}$. Suppose we observe data

$5.691, 5.107, 4.644, 4.970, 5.218, 4.418, 5.021, 5.239, 4.752, 6.071$.

Compute the 95% $z$ and $t$ confidence intervals for $\mu$, respectively, and compare their lengths. You need to include your mathematical details as well as your R code and output.

(b) Data: $X \sim Bin(50, \theta)$. The binomial $z$ confidence interval is $\hat{\theta} \pm z_{\alpha/2} \sqrt{\hat{\theta}(1-\hat{\theta})/n}$, where $\hat{\theta} = X/n$. Suppose we observe $x = 15$. Compute the 95% $z$ confidence interval for $\theta$. You need to include your mathematical details as well as your R code and output.

(c) Data; $X_1, \cdots, X_n \sim Poisson(\theta)$. The Poisson $z$ confidence interval is $\bar{x} \pm z_{\alpha/2} \sqrt{\bar{x}/n}$. Suppose we observe data

$21, 23, 25, 12, 19$.

Compute the 95% $z$ confidence interval for $\theta$. You need to include your mathematical details as well as your R code and output.

2. $z$ and $t$ confidence intervals are both popular. In particular, let $X_1, \cdots, X_n \sim iid N(\mu, \sigma^2)$, where both $\mu$ and $\sigma^2$ are unknown. Then, the $1 - \alpha$ level $z$ confidence interval for $\mu$ is $\bar{x} \pm z_{\alpha/2} s/\sqrt{n}$, and the $1 - \alpha$ level $t$ confidence interval for $\mu$ is $\bar{x} \pm t_{\alpha/2, n-1} s/\sqrt{n}$. Use simulation with $10^4$ replications to evaluate the coverage probability and the length of the 95% $z$ and $t$ confidence intervals for $\mu$ when $n = 5, 10, 20, 50, 100$ with $\mu = 0, 1, 2, 3, 4, 5$ and fixed $\sigma^2 = 1$, respectively. You need to provide a simple description of your simulation, a summary of your findings, and your R code and output.

3. $z$ confidence interval is often used in binomial data for success/failure experiments. In particular, suppose the success/failure experiment is repeated $n$ times, such that the total number of successes $X \sim Bin(n, \theta)$, where $\theta$ is the probability of success in each experiment. Then, the $1 - \alpha$ level $z$ confidence interval for $\theta$ is $\hat{\theta} \pm z_{\alpha/2} \sqrt{x/n}$. Use simulation with $10^4$ replications to evaluate the coverage probability of the 95% $z$ confidence interval for $\theta$ when $\theta$ is close 0, where you choose $\theta = 0.01, 0.02, 0.05, 0.1$ with $n = 10, 20, 50, 100, 200, 500, 1000$, respectively. You need to provide a simple description of your simulation, a summary of your findings, and your R code and output.