Please use R to do all of the following problems. Submit your homework by a WORD or PDF document with main results and the corresponding code.

1. (Compare two estimators by MSE in uniform distributions). Let $X_1, \cdots, X_n \sim iid \ Uniform([0, \theta])$, where we can estimate $\theta$ by $\hat{\theta} = \max\{X_1, X_2, \cdots, X_n\}$ and $\hat{\theta} = 2\bar{X}$. We can use simulations to compare the two estimators. To do this, you need
   (a) Generate $X_1, \cdots, X_n$ from $Uniform([0, \theta])$.
   (b) Compute the maximum of $X_1, \cdots, X_n$ which is $\hat{\theta}$. Compute $\bar{X}$ the average of $X_1, \cdots, X_n$, and let $\tilde{\theta} = 2\bar{X}$.
   (c) Calculate $(\hat{\theta} - \theta)^2$ and $(\tilde{\theta} - \theta)^2$, respectively.
   (d) Repeat (a)–(c) $10^5$ times, and report the average of $(\hat{\theta} - \theta)^2$ and $(\tilde{\theta} - \theta)^2$, which are treated as the MSE of $\hat{\theta}$ and $\tilde{\theta}$, respectively.

   Please choose $n = 10, 20, 50, 100$ and $\theta = 1, 2, 5, 10$. The better one should have a lower MSE value. Summarize your findings.

2. (Mean and median in $t$-distributions). If $X_1, \cdots, X_n \sim iid \ mu + t_1$, then there are two estimators for $\mu$. The first is $\hat{\mu}_1 = \text{median}(X_1, \cdots, X_n)$. The second is $\hat{\mu}_2 = \text{mean}(X_1, \cdots, X_n)$. You can use MSE to compare the two estimators by the following.
   (a) Generate data by
   \[
   x <- \text{mu} + \text{rt}(n, 1)
   \]
   (b) Calculate the median and the mean of the data, denoted by $\hat{\mu}_1$ and $\hat{\mu}_2$, respectively.
   (c) Derive $(\hat{\mu}_1 - \mu)^2$ and $(\hat{\mu}_2 - \mu)^2$.
   (d) Repeat (a)–(c) $10^5$ times. Report MSEs of $\hat{\mu}_1$ and $\hat{\mu}_2$ by taking the average of your those for $\hat{\mu}_1$ and $\hat{\mu}_2$, respectively.

   Please choose $\mu = 0, 2, 5, 10$ and $n = 10, 20, 50, 100$. The better one should have a lower MSE value. Summarize your findings.

3. (Poisson expected value and variance). If $X \sim \text{Poisson}(\lambda)$, then $E(X) = V(X) = \lambda$. Thus, we can use either sample mean or sample variance to estimate $\lambda$. In particular, suppose we have data $X_1, \cdots, X_n \sim iid \ Poisson(\lambda)$. Then, we can calculate $\bar{X}$ and $S^2$ based on the data. You can use MSE to compare the two estimators by the following
   - Generate $n$ data from $\text{Poisson}(\lambda)$.
   - Calculate $\bar{X}$ and $S^2$.
   - Derive $(\bar{X} - \lambda)^2$ and $(S^2 - \lambda)^2$.
   - Repeat (a)–(c) $10^5$ times. Take the average of those, respectively. Then, you have the MSE of $\bar{X}$ and $S^2$. The lower one is better.

   Please choose $\lambda = 1, 2, 5, 10$ and $n = 10, 20, 50, 100$, respectively. Provide your findings.