Homework # 4, Stat355, Spring 2021

Please use R to do all of the following problems. Submit your homework by a WORD or PDF document with main results and the corresponding code. You need to summarize your result but not just simply copy and paste your answers.

1. Let \( f(x) \) be a continuous function of \( x \). Suppose that we want to find root \( x^* \) such that \( f(x^*) = 0 \). We can do like this. Find \( x_1 \) and \( x_2 \) such that one of \( f(x_1) \) and \( f(x_2) \) is positive and another of them is negative. Let \( x_0 = (x_1 + x_2)/2 \). If the sign of \( f(x_0) \) is the same as the sign of \( f(x_1) \), then the root is in \( (x_0, x_2) \) and set \( x_1 = x_0 \); otherwise, the root is within \( (x_1, x_0) \) and set \( x_2 = x_0 \). Iterate this. You can find the root. Please use the idea to find root of the following functions. The difference between the solution you provide and the true \( x^* \) should not be greater than 0.0001 in absolute value.

   (a) Let \( f(x) = e^x - 3x \). Find root \( x^* \) of \( f(x) \) for \( x^* \in (0, 1) \).

   (b) Let \( f(x) = x^3 - 3x - 6 \). Find root \( x^* \) of \( f(x) \) for \( x^* \in (2, 3) \).

2. We can also use computer to numerically calculate the maximum of a function. Let \( f(x) = -x^4 + 4x^3 + 6x^2 - 5x - 7 \). Then, \( f'(x) = -4x^3 + 12x^2 + 12x - 5 \) and \( f''(x) = -12x^2 + 24x + 12 \). Let \( x_0 \) be an initial guess of the maximizer \( x^* \) of \( f(x) \). Then, we can update \( x_0 \) by \( x_0 \leftarrow x_0 - f'(x_0)/f''(x_0) \). For this particular function, you need to do: (a) using a plot to find that \( x^* \) is between 3.0 and 4.0; (b) choose an initial value of \( x_0 \in [3.0, 4.0] \) and use the formula to update \( x_0 \). You need to do the iteration at least 5 times. To ensure the final answer is almost the maximum. You need to check the value of \( f'(x_0) \) in your iterations. Use this idea to find the maximizer of \( f(x) \) with at least 4 different options of \( x_0 \) in \([3.0, 4.0]\). Report each step of your computation with the values of \( f(x) \), \( f'(x) \), and \( f''(x) \) by a table.

3. Let \( X_1, \ldots, X_n \) be iid (identically independently distributed) of the uniform distributions on \([0, 1]\), denoted by \( X_1, \ldots, X_n \sim iid \ U([0, 1]) \). Let \( X_{(1)} \) be the minimum and \( X_{(n)} \) be the maximum of the data. Then, it is known that \( \sqrt{n}X_{(1)} \) and \( \sqrt{n}(1 - X_{(n)}) \) can be approximated by some distribution. You can use computer simulation to confirm. Generate \( n \) observations from the uniform distribution on \([0, 1]\) with \( n = 10^2, 10^3, 10^4, 10^5 \), and \( 10^6 \) respectively. Derive the minimum and the maximum the data, denoted by \( x_{(1)} \) and \( x_{(n)} \), respectively. Repeat the procedure 10000 times for each selected \( n \). Report the averages of \( nx_{(1)} \) and \( n(x_{(n)}) \) with respect to \( n \) by a table. Simply draw your conclusion based on your results.