Supplement to "Comments on the Neyman-Fisher Controversy and its Consequences"

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APPENDIX A: DETAILS OF ANOVA CALCULATIONS

This supplement contains our reworking of Neyman's calculations. It is important to note that, although our proofs may not be technically elegant, they are designed to reveal explicitly the errors of Neyman (1935).

A.1 Randomized Complete Block Designs

Consider N blocks and T treatments, with each block having T experimental units, and treatments randomized to experimental units independently across blocks. We define

$$W_{ij}(t) = \begin{cases} 1 & \text{if unit } j \text{ in block } i \text{ is assigned treatment } t, \\ 0 & \text{otherwise.} \end{cases}$$

Following Neyman (1935), the potential outcome of unit $j = 1, \ldots, T$ in block $i = 1, \ldots, N$ under treatment $t = 1, \ldots, T$ is

$$x_{ij}(t) = X_{ij}(t) + \epsilon_{ij}(t),$$

where $X_{ij}(t) \in \mathbb{R}$ is an unknown constant and $\epsilon_{ij}(t) \sim [0, \sigma_\epsilon^2]$ are iid and independent of treatment indicators $\mathbf{W} = \{W_{ij}(t)\}$. The potential outcomes are decomposed into

$$x_{ij}(t) = \bar{X}_{\cdot\cdot}(t) + B_i(t) + \eta_{ij}(t) + \epsilon_{ij}(t),$$

where

$$B_i(t) = \bar{X}_{i\cdot}(t) - \bar{X}_{\cdot\cdot}(t),$$
$$\eta_{ij}(t) = X_{ij}(t) - \bar{X}_{i\cdot}(t).$$

Define $y_i(t)$ as the observed response of the unit assigned treatment t in block i,

$$y_i(t) = \sum_{j=1}^{T} W_{ij}(t) x_{ij}(t),$$

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and $\bar{y}.(t)$ as the observed average response for units assigned treatment t,

$$\bar{y}.(t) = \frac{1}{N}\sum_{i=1}^{N} y_i(t).$$

We see that

$$\mathbb{E}\{\bar{y}.(t)\} = \mathbb{E}[\mathbb{E}\{\bar{y}.(t)|\mathbf{W}\}] = \mathbb{E}\left\{\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)X_{ij}(t)\right\} = \bar{X}..(t),$$

so $\bar{y}.(t)-\bar{y}.(t')$ is unbiased for $\bar{X}..(t)-\bar{X}..(t')$. We proceed to calculate the variance of this statistic as

$$\mathrm{Var}\{\bar{y}.(t)-\bar{y}.(t')\} = \frac{2\sigma_\epsilon^2 + \sigma_\eta^2(t) + \sigma_\eta^2(t')}{N} + \frac{2r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}}{N(T-1)},$$

where we define

$$\sigma_\eta^2(t) = \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{T}\eta_{ij}(t)^2,$$

$$r(t,t') = \frac{\sum_{i=1}^{N}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t')}{NT\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}}.$$

First, we calculate $\mathrm{Var}\{\bar{y}.(t)\} = \{\sigma_\epsilon^2 + \sigma_\eta^2(t)\}/N$. Note that for $j \neq j',\ i \neq i'$,

$$\mathrm{Cov}\{W_{ij}(t), W_{ij'}(t)\} = \mathbb{E}\{W_{ij}(t)W_{ij'}(t)\} - \mathbb{E}\{W_{ij}(t)\}\mathbb{E}\{W_{ij'}(t)\} = -\frac{1}{T^2},$$
$$\mathrm{Cov}\{W_{ij}(t), W_{i'j}(t)\} = \mathrm{Cov}\{W_{ij}(t), W_{i'j'}(t)\} = 0.$$

Then

$$\begin{aligned}
\mathrm{Var}\{\bar{y}.(t)\} &= \mathbb{E}[\mathrm{Var}\{\bar{y}.(t)|\mathbf{W}\}] + \mathrm{Var}[\mathbb{E}\{\bar{y}.(t)|\mathbf{W}\}] \\
&= \mathbb{E}\left\{\frac{1}{N^2}\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)^2\sigma_\epsilon^2\right\} \\
&\quad + \mathrm{Var}\left[\frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)\{\bar{X}..(t) + B_i(t) + \eta_{ij}(t)\}\right] \\
&= \frac{\sigma_\epsilon^2}{N} + \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}\left\{\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t)\right\} \\
&= \frac{\sigma_\epsilon^2}{N} + \frac{1}{N^2}\sum_{i=1}^{N}\left\{\sum_{j=1}^{T}\frac{1}{T}\left(1-\frac{1}{T}\right)\eta_{ij}(t)^2 + \sum_{j\neq j'}\left(-\frac{1}{T^2}\right)\eta_{ij}(t)\eta_{ij'}(t)\right\} \\
&= \frac{\sigma_\epsilon^2}{N} + \frac{1}{N^2 T}\sum_{i=1}^{N}\sum_{j=1}^{T}\eta_{ij}(t)^2 - \frac{1}{N^2T^2}\sum_{i=1}^{N}\left\{\sum_{j=1}^{T}\eta_{ij}(t)\right\}^2 \\
&= \frac{\sigma_\epsilon^2 + \sigma_\eta^2(t)}{N}.
\end{aligned}$$

To find $\mathrm{Cov}\{\bar{y}.(t), \bar{y}.(t')\} = -r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}/\{N(T-1)\}$, note that

$$\mathrm{Cov}\{\bar{y}.(t), \bar{y}.(t')|\mathbf{W}\} = \frac{1}{N^2}\mathrm{Cov}\left\{\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)\epsilon_{ij}(t), \sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t')\epsilon_{ij}(t')|\mathbf{W}\right\} = 0,$$

$$\begin{aligned}\mathbb{E}\{\bar{y}.(t)|\mathbf{W}\} &= \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)\{\bar{X}..(t) + B_i(t) + \eta_{ij}(t)\} \\ &= \bar{X}..(t) + \frac{1}{N}\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t).\end{aligned}$$

Then

$$\begin{aligned}\mathrm{Cov}\{\bar{y}.(t), \bar{y}.(t')\} &= \mathbb{E}[\mathrm{Cov}\{\bar{y}.(t), \bar{y}.(t')|\mathbf{W}\}] + \mathrm{Cov}[\mathbb{E}\{\bar{y}.(t)|\mathbf{W}\}, \mathbb{E}\{\bar{y}.(t')|\mathbf{W}\}] \\ &= \frac{1}{N^2}\mathrm{Cov}\left\{\sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t), \sum_{i=1}^{N}\sum_{j=1}^{T} W_{ij}(t')\eta_{ij}(t')\right\} \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Cov}\left\{\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t), \sum_{j=1}^{T} W_{ij}(t')\eta_{ij}(t')\right\} \\ &= \frac{1}{N^2}\sum_{i=1}^{N}\left[\sum_{j=1}^{T}\left(-\frac{1}{T^2}\right)\eta_{ij}(t)\eta_{ij}(t') + \sum_{j\neq j'}\left\{\frac{1}{T(T-1)} - \frac{1}{T^2}\right\}\eta_{ij}(t)\eta_{ij'}(t')\right] \\ &= -\frac{r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}}{N(T-1)}.\end{aligned}$$

Thus

$$\mathrm{Var}\{\bar{y}.(t) - \bar{y}.(t')\} = \frac{2\sigma_\epsilon^2 + \sigma_\eta^2(t) + \sigma_\eta^2(t')}{N} + \frac{2r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}}{N(T-1)}.$$

We now calculate expectations of sums of squares, starting with residual sum of squares

$$(N-1)(T-1)S_0^2 = \sum_{i=1}^{N}\sum_{t=1}^{T}\{y_i(t) - \bar{y}.(t) - \bar{y}_i(\cdot) + \bar{y}.(\cdot)\}^2,$$

where we rewrite

$$\{y_i(t)-\bar{y}.(t)-\bar{y}_i(\cdot)+\bar{y}.(\cdot)\}^2 = \{y_i(t)-\bar{y}.(t)\}^2+\{\bar{y}_i(\cdot)-\bar{y}.(\cdot)\}^2-2\{y_i(t)-\bar{y}.(t)\}\{\bar{y}_i(\cdot)-\bar{y}.(\cdot)\}.$$

As

$$\mathbb{E}\{y_i(t) - \bar{y}.(t)\} = \frac{1}{T}\sum_{j=1}^{T} X_{ij}(t) - \frac{1}{NT}\sum_{i=1}^{N}\sum_{j=1}^{T} X_{ij}(t) = B_i(t),$$

we have

$$\mathbb{E}\{y_i(t) - \bar{y}.(t)\}^2 = \mathrm{Var}\{y_i(t)\} + \mathrm{Var}\{\bar{y}.(t)\} - 2\mathrm{Cov}\{y_i(t), \bar{y}.(t)\} + B_i(t)^2.$$

Also,

$$\begin{aligned}\mathrm{Var}\{y_i(t)\} &= \mathbb{E}\left\{\sum_{j=1}^{T} W_{ij}(t)^2\sigma_\epsilon^2\right\} + \mathrm{Var}\left[\sum_{j=1}^{T} W_{ij}(t)\{\bar{X}_{\cdot\cdot}(t) + B_i(t) + \eta_{ij}(t)\}\right] \\ &= \sigma_\epsilon^2 + \frac{1}{T}\sum_{j=1}^{T}\eta_{ij}(t)^2.\end{aligned}$$

For now, we write

$$\mathbb{E}\{y_i(t) - \bar{y}_\cdot(t)\}^2 = \sigma_\epsilon^2 + \frac{1}{T}\sum_{j=1}^{T}\eta_{ij}(t)^2 + \frac{\sigma_\epsilon^2 + \sigma_\eta^2(t)}{N} - 2\mathrm{Cov}\{y_i(t), \bar{y}_\cdot(t)\} + B_i(t)^2.$$

From above,

$$\mathbb{E}\{\bar{y}_i(\cdot) - \bar{y}_\cdot(\cdot)\} = \frac{1}{T}\sum_{t=1}^{T}\mathbb{E}\{y_i(t) - \bar{y}_\cdot(t)\} = \bar{B}_i(\cdot),$$

and we write

$$\mathbb{E}\{\bar{y}_i(\cdot) - \bar{y}_\cdot(\cdot)\}^2 = \mathrm{Var}\{\bar{y}_i(\cdot)\} + \mathrm{Var}\{\bar{y}_\cdot(\cdot)\} - 2\mathrm{Cov}\{\bar{y}_i(\cdot), \bar{y}_\cdot(\cdot)\} + \bar{B}_i(\cdot)^2.$$

Finally,

$$\begin{aligned}\mathbb{E}[\{y_i(t) - \bar{y}_\cdot(t)\}\{\bar{y}_i(\cdot) - \bar{y}_\cdot(\cdot)\}] = {} & \mathrm{Cov}\{y_i(t), \bar{y}_i(\cdot)\} - \mathrm{Cov}\{y_i(t), \bar{y}_\cdot(\cdot)\} - \mathrm{Cov}\{\bar{y}_\cdot(t), \bar{y}_i(\cdot)\} \\ & + \mathrm{Cov}\{\bar{y}_\cdot(t), \bar{y}_\cdot(\cdot)\} + B_i(t)\bar{B}_i(\cdot).\end{aligned}$$

To simplify the remaining calculations, we evaluate $\mathrm{Cov}\{y_i(t), y_i(t')\}$ and $\mathrm{Var}\{\bar{y}_\cdot(\cdot)\}$.

$$\begin{aligned}\mathrm{Cov}\{y_i(t), y_i(t')\} &= \mathrm{Cov}\left\{\sum_{j=1}^{T} W_{ij}(t)X_{ij}(t), \sum_{j=1}^{T} W_{ij}(t')X_{ij}(t')\right\} \\ &= \mathrm{Cov}\left\{\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t), \sum_{j=1}^{T} W_{ij}(t')\eta_{ij}(t')\right\} \\ &= \sum_{j=1}^{T}\mathrm{Cov}\{W_{ij}(t)\eta_{ij}(t), W_{ij}(t')\eta_{ij}(t')\} \\ &\quad + \sum_{j\neq j'}\mathrm{Cov}\{W_{ij}(t)\eta_{ij}(t), W_{ij'}(t')\eta_{ij'}(t')\} \\ &= -\frac{1}{T^2}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t') + \left\{\frac{1}{T(T-1)} - \frac{1}{T^2}\right\}\sum_{j\neq j'}\eta_{ij}(t)\eta_{ij'}(t') \\ &= -\frac{1}{T(T-1)}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t') \\ &\quad + \left\{\frac{1}{T(T-1)} - \frac{1}{T^2}\right\}\left\{\sum_{j=1}^{T}\eta_{ij}(t)\right\}\left\{\sum_{j=1}^{T}\eta_{ij}(t')\right\} \\ &= -\frac{1}{T(T-1)}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t').\end{aligned}$$

As treatments are assigned independently across blocks,

$$
\begin{aligned}
\mathrm{Var}\{\bar{y}.(\cdot)\} &= \frac{1}{N^2}\sum_{i=1}^{N}\mathrm{Var}\{\bar{y}_i(\cdot)\} \\
&= \left(\frac{1}{NT}\right)^2\sum_{i=1}^{N}\left[\sum_{t=1}^{T}\mathrm{Var}\{y_i(t)\} + \sum_{t\neq t'}\mathrm{Cov}\{y_i(t), y_i(t')\}\right] \\
&= \left(\frac{1}{NT}\right)^2\sum_{i=1}^{N}\left[\sum_{t=1}^{T}\left\{\sigma_\epsilon^2 + \frac{1}{T}\sum_{j=1}^{T}\eta_{ij}(t)^2\right\} - \frac{1}{T(T-1)}\sum_{t\neq t'}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t')\right] \\
&= \left(\frac{1}{NT}\right)^2\left\{NT\sigma_\epsilon^2 + \frac{1}{T}\sum_{i=1}^{N}\sum_{j=1}^{T}\sum_{t=1}^{T}\eta_{ij}(t)^2 - \frac{1}{T(T-1)}\sum_{t\neq t'}\sum_{i=1}^{N}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t')\right\} \\
&= \frac{\sigma_\epsilon^2}{NT} + \frac{1}{NT^2}\sum_{t=1}^{T}\sigma_\eta^2(t) - \frac{1}{NT^2(T-1)}\sum_{t\neq t'}r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}.
\end{aligned}
$$

Finally, we note that

$$
\sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t)^2 - \bar{B}_i(\cdot)^2\} = \sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t) - \bar{B}_i(\cdot)\}^2.
$$

We use all these results for the following simplifications:

$$
\begin{aligned}
\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\{y_i(t) - \bar{y}.(t)\}^2 = (N+1)T\sigma_\epsilon^2 + (N+1)\sum_{t=1}^{T}\sigma_\eta^2(t) - 2N\sum_{t=1}^{T}\mathrm{Var}\{\bar{y}.(t)\} \\
+ \sum_{i=1}^{N}\sum_{t=1}^{T}B_i(t)^2,
\end{aligned}
$$

$$
\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}\{\bar{y}_i(\cdot) - \bar{y}.(\cdot)\}^2 = N(N-1)T\mathrm{Var}\{\bar{y}.(\cdot)\} + \sum_{i=1}^{N}\sum_{t=1}^{T}\bar{B}_i(\cdot)^2,
$$

$$
-2\sum_{i=1}^{N}\sum_{t=1}^{T}\mathbb{E}[\{y_i(t)-\bar{y}.(t)\}\{\bar{y}_i(\cdot)-\bar{y}.(\cdot)\}] = -2N(N-1)T\mathrm{Var}\{\bar{y}.(\cdot)\} - 2\sum_{i=1}^{N}\sum_{t=1}^{T}\bar{B}_i(\cdot)^2.
$$

Combining these terms, we obtain

$$
\begin{aligned}
\mathbb{E}\{(N-1)(T-1)S_0^2\} &= (N+1)T\sigma_\epsilon^2 + (N+1)\sum_{t=1}^{T}\sigma_\eta^2(t) - N(N-1)T\mathrm{Var}\{\bar{y}.(\cdot)\} \\
&\quad - 2N\sum_{t=1}^{T}\mathrm{Var}\{\bar{y}.(t)\} + \sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t)^2 - \bar{B}_i(\cdot)^2\} \\
&= (N-1)T\sigma_\epsilon^2 + (N-1)\sum_{t=1}^{T}\sigma_\eta^2(t) - N(N-1)T\mathrm{Var}\{\bar{y}.(\cdot)\} \\
&\quad + \sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t) - \bar{B}_i(\cdot)\}^2,
\end{aligned}
$$

so that

$$\begin{aligned}\mathbb{E}\{(N-1)(T-1)S_0^2\} = (N-1)(T-1)\sigma_\epsilon^2 + (N-1)\left(1-\frac{1}{T}\right)\sum_{t=1}^{T}\sigma_\eta^2(t) \\ + \frac{N-1}{T(T-1)}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')} \\ + \sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t) - \bar{B}_i(\cdot)\}^2.\end{aligned}$$

This is the correct expression for the expected residual sum of squares. As

$$\sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t) - \bar{B}_i(\cdot)\}^2 \neq 0$$

in general, this differs from the one given by Neyman, which we ow derive.

From pages 147-148 of his appendix, we see that Neyman calculates

$$\mathbb{E}\{(N-1)(T-1)S_0^2\} = (N-1)(T-1)\mathbb{E}(S_0'^2 + S_0''^2),$$

where we define

$$S_0'^2 = \sum_{i=1}^{N}\sum_{t=1}^{T}\{\eta_i(t) - \bar{\eta}.(t) - \bar{\eta}_i(\cdot) + \bar{\eta}.(\cdot)\}^2,$$

$$S_0''^2 = \sum_{i=1}^{N}\sum_{t=1}^{T}\{\epsilon_i(t) - \bar{\epsilon}.(t) - \bar{\epsilon}_i(\cdot) + \bar{\epsilon}.(\cdot)\}^2,$$

$$\eta_i(t) = \sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t), \quad \epsilon_i(t) = \sum_{j=1}^{T} W_{ij}(t)\epsilon_{ij}(t).$$

We have from equations (21) – (24) in Neyman's appendix that

$$\begin{aligned}\mathbb{E}\left\{(N-1)(T-1)\sum_{i=1}^{N}\sum_{t=1}^{T}\eta_i(t)^2\right\} &= \frac{(N-1)(T-1)}{T}\sum_{i=1}^{N}\sum_{j=1}^{T}\sum_{t=1}^{T}\eta_{ij}(t)^2 \\ &= N(N-1)(T-1)\sum_{t=1}^{T}\sigma_\eta^2(t),\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left\{-(N-1)\sum_{t\neq t'}\sum_{i=1}^{N}\eta_i(t)\eta_i(t')\right\} &= -\frac{N-1}{T(T-1)}\sum_{t\neq t'}\sum_{i=1}^{N}\sum_{j\neq j'}\eta_{ij}(t)\eta_{ij'}(t') \\ &= \frac{N-1}{T(T-1)}\sum_{t\neq t'}\sum_{i=1}^{N}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t') \\ &\quad - \frac{N-1}{T(T-1)}\sum_{t\neq t'}\sum_{i=1}^{N}\left\{\sum_{j=1}^{T}\eta_{ij}(t)\right\}\left\{\sum_{j=1}^{T}\eta_{ij}(t')\right\} \\ &= \frac{N(N-1)}{T-1}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}.\end{aligned}$$

Using Neyman's notation

$$\mathbb{E}(S_0'^2) = \frac{1}{T}\sum_{t=1}^{T}\sigma_\eta^2(t) + \frac{1}{T(T-1)^2}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')},$$

$$\mathbb{E}(S_0''^2) = \sigma_\epsilon^2.$$

Thus, Neyman obtains

$$\mathbb{E}(S_0^2) = \sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\sigma_\eta^2(t) + \frac{1}{T(T-1)^2}\sum_{t\neq t} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}.$$

We see that Neyman's result for the expected mean residual sum of squares is generally less than the correct expression. In fact, Neyman's error occurs in equation (17) on page 147 of his appendix. His final result is missing the term $\sum_{i=1}^{N}\sum_{t=1}^{T}\{B_i(t) - \bar{B}_i(\cdot)\}^2/\{(N-1)(T-1)\}$.

We finally calculate the expectation of the mean treatment sum of squares,

$$S_1^2 = \frac{N}{T-1}\sum_{t=1}^{T}\left\{\bar{y}.(t) - \bar{y}.(\cdot)\right\}^2.$$

Now $\mathbb{E}\{\bar{y}.(t)\} = \bar{X}..(t),\ \mathbb{E}\{\bar{y}.(\cdot)\} = \bar{X}..(\cdot)$, so that

$$\begin{aligned}
\mathbb{E}(S_1^2) &= \frac{N}{T-1}\sum_{t=1}^{T}\left[\mathrm{Var}\{\bar{y}.(t)\} + \mathrm{Var}\{\bar{y}.(\cdot)\} - 2\mathrm{Cov}\{\bar{y}.(t),\bar{y}.(\cdot)\} + \{\bar{X}..(t) - \bar{X}..(\cdot)\}^2\right] \\
&= \frac{N}{T-1}\left[\sum_{t=1}^{T}\mathrm{Var}\{\bar{y}.(t)\} + T\mathrm{Var}\{\bar{y}.(\cdot)\} - 2T\mathrm{Var}\{\bar{y}.(\cdot)\} + \sum_{t=1}^{T}\{\bar{X}..(t) - \bar{X}..(\cdot)\}^2\right] \\
&= \sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\sigma_\eta^2(t) + \frac{1}{T(T-1)^2}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')} + \frac{N}{T-1}\sum_{t=1}^{T}\{\bar{X}..(t) - \bar{X}..(\cdot)\}^2,
\end{aligned}$$

which corresponds to Neyman's result.

A.2 Latin Square Designs

We now consider $T \times T$ LSs, with rows and columns denoting levels of two blocking factors. Define

$$W_{ij}(t) = \begin{cases} 1 & \text{if the unit in row } i,\ \text{column } j,\ \text{is assigned treatment } t, \\ 0 & \text{otherwise.} \end{cases}$$

The potential outcome of unit (i,j) under treatment t is

$$x_{ij}(t) = X_{ij}(t) + \epsilon_{ij}(t),$$

with $X_{ij}(t) \in \mathbb{R}$ an unknown constant and $\epsilon_{ij}(t) \sim [0, \sigma_\epsilon^2]$ iid and independent of the $W_{ij}(t)$. These are decomposed into

$$x_{ij}(t) = \bar{X}..(t) + R_i(t) + C_j(t) + \eta_{ij}(t) + \epsilon_{ij}(t),$$

where

$$R_i(t) = \bar{X}_{i\cdot}(t) - \bar{X}_{\cdot\cdot}(t),$$
$$C_j(t) = \bar{X}_{\cdot j}(t) - \bar{X}_{\cdot\cdot}(t),$$
$$\eta_{ij}(t) = X_{ij}(t) - \bar{X}_{i\cdot}(t) - \bar{X}_{\cdot j}(t) + \bar{X}_{\cdot\cdot}(t).$$

Define $\bar{x}^o_{\cdot\cdot}(t)$ as the observed average response for units assigned treatment t,

$$\bar{x}^o_{\cdot\cdot}(t) = \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)x_{ij}(t).$$

To calculate expectations for the LS, we use the following probabilities, which are proven in the next subsection:

$$\Pr\{W_{ij}(t) = 1\} = \frac{1}{T},$$

$$\Pr\{W_{ij}(t) = W_{ij'}(t) = 1\} = \Pr\{W_{ij}(t) = W_{i'j}(t) = 1\} = 0,$$

$$\begin{aligned}\Pr\{W_{ij}(t) = W_{ij'}(t') = 1\} &= \Pr\{W_{ij}(t) = W_{i'j}(t') = 1\} \\ &= \Pr\{W_{ij}(t) = W_{i'j'}(t) = 1\} \\ &= \frac{1}{T(T-1)},\end{aligned}$$

$$\Pr\{W_{ij}(t) = W_{i'j'}(t') = 1\} = \frac{T-2}{T(T-1)^2}.$$

Again, $\mathbb{E}\{\bar{x}^o_{\cdot\cdot}(t)\} = \bar{X}_{\cdot\cdot}(t)$. We next calculate $\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)\} = \sigma^2_\epsilon/T + \sigma^2_\eta(t)/(T-1)$, where

$$\sigma^2_\eta(t) = \frac{1}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T}\eta_{ij}(t)^2.$$

As $\sum_{i=1}^T R_i(t) = \sum_{j=1}^T C_j(t) = 0$ and $\sum_{i=1}^T W_{ij}(t) = \sum_{j=1}^T W_{ij}(t) = 1$,

$$\bar{x}^o_{\cdot\cdot}(t) = \bar{X}_{\cdot\cdot}(t) + \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t) + \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)\epsilon_{ij}(t).$$

By conditioning on $\mathbf{W}$,

$$\begin{aligned}\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)\} &= \mathbb{E}\left\{\frac{\sigma^2_\epsilon}{T^2}\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)^2\right\} + \mathrm{Var}\left\{\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t)\right\} \\ &= \frac{\sigma^2_\epsilon}{T} + \frac{1}{T^2}\mathrm{Var}\left\{\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t)\right\}.\end{aligned}$$

We see that

$$\begin{aligned}\operatorname{Var}\left\{\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t)\right\} &= \sum_{i=1}^{T}\sum_{j=1}^{T}\left(\frac{1}{T}-\frac{1}{T^2}\right)\eta_{ij}(t)^2 + \sum_{i=1}^{T}\sum_{j\neq j'}\left(-\frac{1}{T^2}\right)\eta_{ij}(t)\eta_{ij'}(t)\\ &\quad+\sum_{i\neq i'}\sum_{j=1}^{T}\left(-\frac{1}{T^2}\right)\eta_{ij}(t)\eta_{i'j}(t)\\ &\quad+\sum_{i\neq i'}\sum_{j\neq j'}\left\{\frac{1}{T(T-1)}-\frac{1}{T^2}\right\}\eta_{ij}(t)\eta_{i'j'}(t)\\ &=\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\eta_{ij}(t)^2-\frac{1}{T^2}\left\{\sum_{i=1}^{T}\sum_{j=1}^{T}\eta_{ij}(t)\right\}^2\\ &\quad+\frac{1}{T(T-1)}\sum_{i\neq i'}\sum_{j\neq j'}\eta_{ij}(t)\eta_{i'j'}(t).\end{aligned}$$

Now $\sum_{i=1}^{T}\eta_{ij}(t)=\sum_{j=1}^{T}\eta_{ij}(t)=0$, and for fixed $i,j\in\{1,\dots,T\}$,

$$\sum_{i'\neq i}\sum_{j'\neq j}\eta_{ij}(t)\eta_{i'j'}(t)=\eta_{ij}(t)\sum_{i'\neq i}\sum_{j'\neq j}\eta_{i'j'}(t)=\eta_{ij}(t)\sum_{i'\neq i}\{-\eta_{i'j}(t)\}=\eta_{ij}(t)^2.$$

Hence

$$\operatorname{Var}\left\{\sum_{i=1}^{T}\sum_{j=1}^{T} W_{ij}(t)\eta_{ij}(t)\right\}=\left\{\frac{1}{T}+\frac{1}{T(T-1)}\right\}\sum_{i=1}^{T}\sum_{j=1}^{T}\eta_{ij}(t)^2=\frac{1}{T-1}\sum_{i=1}^{T}\sum_{j=1}^{T}\eta_{ij}(t)^2,$$

and so $\operatorname{Var}\{\bar{x}^{o}_{\cdot\cdot}(t)\}=\sigma_\epsilon^2/T+\sigma_\eta^2(t)/(T-1)$.

We now calculate $\operatorname{Cov}\{\bar{x}^{o}_{\cdot\cdot}(t),\bar{x}^{o}_{\cdot\cdot}(t')\}=-r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}/(T-1)^2$, where

$$r(t,t')=\frac{\sum_{i=1}^{T}\sum_{j=1}^{T}\eta_{ij}(t)\eta_{ij}(t')}{T^2\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}}.$$

We see that

$$\operatorname{Cov}\{\bar{x}^{o}_{\cdot\cdot}(t),\bar{x}^{o}_{\cdot\cdot}(t')|\mathbf{W}\}=\frac{1}{T^2}\operatorname{Cov}\left\{\sum_{i=1}^{T}\sum_{j=1}^{T}W_{ij}(t)\epsilon_{ij}(t),\sum_{i=1}^{T}\sum_{j=1}^{T}W_{ij}(t')\epsilon_{ij}(t')|\mathbf{W}\right\}=0.$$

As

$$\mathbb{E}\{\bar{x}^{o}_{\cdot\cdot}(t)|\mathbf{W}\}=\bar{X}_{\cdot\cdot}(t)+\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}W_{ij}(t)\eta_{ij}(t),$$

we have

$$\begin{aligned}
\mathrm{Cov}[\mathbb{E}\{\bar{x}^o_{\cdot\cdot}(t)|\mathbf{W}\}, \mathbb{E}\{\bar{x}^o_{\cdot\cdot}(t')|\mathbf{W}\}] &= \frac{1}{T^2}\mathrm{Cov}\left\{\sum_{i=1}^T\sum_{j=1}^T W_{ij}(t)\eta_{ij}(t), \sum_{i=1}^T\sum_{j=1}^T W_{ij}(t')\eta_{ij}(t')\right\} \\
&= \frac{1}{T^2}\sum_{i=1}^T\sum_{j=1}^T\left(-\frac{1}{T^2}\right)\eta_{ij}(t)\eta_{ij}(t') \\
&\quad + \frac{1}{T^2}\sum_{i=1}^T\sum_{j\neq j'}\left\{\frac{1}{T(T-1)} - \frac{1}{T^2}\right\}\eta_{ij}(t)\eta_{ij'}(t') \\
&\quad + \frac{1}{T^2}\sum_{i\neq i'}\sum_{j=1}^T\left\{\frac{1}{T(T-1)} - \frac{1}{T^2}\right\}\eta_{ij}(t)\eta_{i'j}(t') \\
&\quad + \frac{1}{T^2}\sum_{i\neq i'}\sum_{j\neq j'}\left\{\frac{T-2}{T(T-1)^2} - \frac{1}{T^2}\right\}\eta_{ij}(t)\eta_{i'j'}(t') \\
&= -\frac{1}{T^4}\left\{\sum_{i=1}^T\sum_{j=1}^T\eta_{ij}(t)\right\}\left\{\sum_{i=1}^T\sum_{j=1}^T\eta_{ij}(t')\right\} \\
&\quad - \frac{1}{T^2(T-1)^2}\sum_{i=1}^T\sum_{j=1}^T\eta_{ij}(t)\eta_{ij}(t') \\
&= -\frac{r(t,t')\sqrt{\sigma^2_\eta(t)\sigma^2_\eta(t')}}{(T-1)^2}.
\end{aligned}$$

We have from all these calculations that

$$\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t) - \bar{x}^o_{\cdot\cdot}(t')\} = \frac{2\sigma^2_\epsilon}{T} + \frac{\sigma^2_\eta(t) + \sigma^2_\eta(t')}{T-1} + \frac{2r(t,t')\sqrt{\sigma^2_\eta(t)\sigma^2_\eta(t')}}{(T-1)^2}.$$

The residual and treatment sums of squares are (respectively)

$$(T-1)(T-2)S^2_0 = \sum_{i=1}^T\sum_{j=1}^T\left\{y_{ij} - \bar{y}_{i\cdot} - \bar{y}_{\cdot j} - \sum_{t=1}^T W_{ij}(t)\bar{x}^o_{\cdot\cdot}(t) + 2\bar{y}_{\cdot\cdot}\right\}^2,$$

$$(T-1)S^2_1 = T\sum_{t=1}^T\{\bar{x}^o_{\cdot\cdot}(t) - \bar{y}_{\cdot\cdot}\}^2,$$

where $y_{ij} = \sum_{t=1}^T W_{ij}(t)x_{ij}(t)$ is the observed response of cell (i,j) and

$$\bar{y}_{i\cdot} = \frac{1}{T}\sum_{j=1}^T y_{ij}, \quad \bar{y}_{\cdot j} = \frac{1}{T}\sum_{i=1}^T y_{ij}, \quad \bar{y}_{\cdot\cdot} = \frac{1}{T}\sum_{j=1}^T \bar{y}_{\cdot j} = \frac{1}{T}\sum_{i=1}^T \bar{y}_{i\cdot} = \frac{1}{T^2}\sum_{i=1}^T\sum_{j=1}^T y_{ij}.$$

We calculate the expected residual sum of squares by subtracting the sum of the expected treatment, column, and row sums of squares from the expected total sum of squares. We see that

$$\mathbb{E}\left\{\sum_{i=1}^{T}\sum_{j=1}^{T}(y_{ij}-\bar{y}_{\cdot\cdot})^2\right\} = \sum_{i=1}^{T}\sum_{j=1}^{T}\mathrm{Var}(y_{ij}-\bar{y}_{\cdot\cdot}) + \sum_{i=1}^{T}\sum_{j=1}^{T}\{\bar{X}_{ij}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2$$

$$= \sum_{i=1}^{T}\sum_{j=1}^{T}\mathrm{Var}(y_{ij}) - T^2\mathrm{Var}(\bar{y}_{\cdot\cdot}) + \sum_{i=1}^{T}\sum_{j=1}^{T}\{\bar{X}_{ij}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2,$$

$$\mathbb{E}\left[T\sum_{t=1}^{T}\{\bar{x}^o_{\cdot\cdot}(t)-\bar{y}_{\cdot\cdot}\}^2\right] = T\sum_{t=1}^{T}\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)-\bar{y}_{\cdot\cdot}\} + T\sum_{t=1}^{T}\{\bar{X}_{\cdot\cdot}(t)-\bar{X}_{\cdot\cdot}(\cdot)\}^2$$

$$= T\sum_{t=1}^{T}\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)\} - T^2\mathrm{Var}(\bar{y}_{\cdot\cdot}) + T\sum_{t=1}^{T}\{\bar{X}_{\cdot\cdot}(t)-\bar{X}_{\cdot\cdot}(\cdot)\}^2,$$

$$\mathbb{E}\left\{T\sum_{j=1}^{T}(\bar{y}_{\cdot j}-\bar{y}_{\cdot\cdot})^2\right\} = T\sum_{j=1}^{T}\mathrm{Var}(\bar{y}_{\cdot j}-\bar{y}_{\cdot\cdot}) + T\sum_{j=1}^{T}\{\bar{X}_{\cdot j}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2$$

$$= T\sum_{j=1}^{T}\mathrm{Var}(\bar{y}_{\cdot j}) - T^2\mathrm{Var}(\bar{y}_{\cdot\cdot}) + T\sum_{j=1}^{T}\{\bar{X}_{\cdot j}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2,$$

$$\mathbb{E}\left\{T\sum_{i=1}^{T}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot})^2\right\} = T\sum_{i=1}^{T}\mathrm{Var}(\bar{y}_{i\cdot}-\bar{y}_{\cdot\cdot}) + T\sum_{i=1}^{T}\{\bar{X}_{i\cdot}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2$$

$$= T\sum_{i=1}^{T}\mathrm{Var}(\bar{y}_{i\cdot}) - T^2\mathrm{Var}(\bar{y}_{\cdot\cdot}) + T\sum_{i=1}^{T}\{\bar{X}_{i\cdot}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2.$$

The expected residual sum of squares is the sum of

$$\sum_{i=1}^{T}\sum_{j=1}^{T}\mathrm{Var}(y_{ij}) - T\sum_{t=1}^{T}\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)\}, \tag{A.1}$$

$$-T\left\{\sum_{i=1}^{T}\mathrm{Var}(\bar{y}_{i\cdot}) + \sum_{j=1}^{T}\mathrm{Var}(\bar{y}_{\cdot j}) - 2T\mathrm{Var}(\bar{y}_{\cdot\cdot})\right\}, \tag{A.2}$$

$$\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{t=1}^{T}[\{\bar{X}_{ij}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2-\{\bar{X}_{\cdot\cdot}(t)-\bar{X}_{\cdot\cdot}(\cdot)\}^2-\{\bar{X}_{\cdot j}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2-\{\bar{X}_{i\cdot}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2], \tag{A.3}$$

and we proceed to evaluate each of these three terms.

First note that (by conditioning on $\mathbf{W}$),

$$\begin{aligned}
\mathrm{Var}(y_{ij}) &= \sigma_\epsilon^2 + \mathrm{Var}\left\{\sum_{t=1}^{T} W_{ij}(t)X_{ij}(t)\right\} \\
&= \sigma_\epsilon^2 + \sum_{t=1}^{T}\left(\frac{1}{T} - \frac{1}{T^2}\right)X_{ij}(t)^2 + \sum_{t\neq t'}\left(-\frac{1}{T^2}\right)X_{ij}(t)X_{ij}(t') \\
&= \sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T} X_{ij}(t)^2 - \bar{X}_{ij}(\cdot)^2 \\
&= \sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\left\{X_{ij}(t) - \bar{X}_{ij}(\cdot)\right\}^2.
\end{aligned}$$

As such, (A.1) can be written as

$$T(T-1)\sigma_\epsilon^2 + \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{t=1}^{T}\{X_{ij}(t) - \bar{X}_{ij}(\cdot)\}^2 - \frac{T}{T-1}\sum_{t=1}^{T}\sigma_\eta^2(t),$$

which we expand as

$$\begin{aligned}
T(T-1)\sigma_\epsilon^2 + T\sum_{t=1}^{T}\{\bar{X}_{\cdot\cdot}(t) - \bar{X}_{\cdot\cdot}(\cdot)\}^2 + \sum_{i=1}^{T}\sum_{t=1}^{T}\{R_i(t) - \bar{R}_i(\cdot)\}^2 + \sum_{j=1}^{T}\sum_{t=1}^{T}\{C_j(t) - \bar{C}_j(\cdot)\}^2 \\
+ \frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{t=1}^{T}\{\eta_{ij}(t) - \bar{\eta}_{ij}(\cdot)\}^2 - \frac{T}{T-1}\sum_{t=1}^{T}\sigma_\eta^2(t).
\end{aligned}$$

To write out the expression for (A.2), note that

$$\mathrm{Var}(\bar{y}_{\cdot\cdot}) = \mathrm{Var}\left\{\frac{1}{T}\sum_{t=1}^{T}\bar{x}^o_{\cdot\cdot}(t)\right\} = \frac{1}{T^2}\sum_{t=1}^{T}\mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)\} + \frac{1}{T^2}\sum_{t\neq t'}\mathrm{Cov}\{\bar{x}^o_{\cdot\cdot}(t), \bar{x}^o_{\cdot\cdot}(t')\},$$

and so

$$2T^2\mathrm{Var}(\bar{y}_{\cdot\cdot}) = 2\sigma_\epsilon^2 + \frac{2}{T-1}\sum_{t=1}^{T}\sigma_\eta^2(t) - \frac{2}{(T-1)^2}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}.$$

By conditioning on $\mathbf{W}$, we have

$$\begin{aligned}
\mathrm{Cov}(y_{ij}, y_{ij'}) &= -\frac{1}{T(T-1)}\sum_{t=1}^{T} X_{ij}(t)X_{ij'}(t) + \left(\frac{1}{T-1}\right)\bar{X}_{ij}(\cdot)\bar{X}_{ij'}(\cdot), \\
\mathrm{Cov}(y_{ij}, y_{i'j}) &= -\frac{1}{T(T-1)}\sum_{t=1}^{T} X_{ij}(t)X_{i'j}(t) + \left(\frac{1}{T-1}\right)\bar{X}_{ij}(\cdot)\bar{X}_{i'j}(\cdot).
\end{aligned}$$

With these relations in mind,

$$
\begin{aligned}
-T\sum_{i=1}^{T}\mathrm{Var}(\bar{y}_{i\cdot}) &= -\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\mathrm{Var}(y_{ij}) - \frac{1}{T}\sum_{i=1}^{T}\sum_{j\neq j'}\mathrm{Cov}(y_{ij}, y_{ij'}) \\
&= -\frac{1}{T}\sum_{i=1}^{T}\sum_{j=1}^{T}\left[\sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\{X_{ij}(t) - \bar{X}_{ij}(\cdot)\}^2\right] \\
&\quad + \frac{1}{T^2(T-1)}\sum_{i=1}^{T}\sum_{j\neq j'}\left\{\sum_{t=1}^{T}X_{ij}(t)X_{ij'}(t) - T\bar{X}_{ij}(\cdot)\bar{X}_{ij'}(\cdot)\right\},
\end{aligned}
$$

and

$$
\sum_{i=1}^{T}\sum_{j\neq j'}\left\{\sum_{t=1}^{T}X_{ij}(t)X_{ij'}(t)\right\} = \sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{t=1}^{T}X_{ij}(t)\{T\bar{X}_{i\cdot}(t) - X_{ij}(t)\},
$$

$$
\sum_{i=1}^{T}\sum_{j\neq j'}\{-T\bar{X}_{ij}(\cdot)\bar{X}_{ij'}(\cdot)\} = -\sum_{i=1}^{T}\sum_{j=1}^{T}T\bar{X}_{ij}(\cdot)\{T\bar{X}_{i\cdot}(\cdot) - \bar{X}_{ij}(\cdot)\}.
$$

By symmetry,

$$
\begin{aligned}
-T\sum_{j=1}^{T}\mathrm{Var}(\bar{y}_{\cdot j}) &= -\frac{1}{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\mathrm{Var}(y_{ij}) - \frac{1}{T}\sum_{j=1}^{T}\sum_{i\neq i'}\mathrm{Cov}(y_{ij}, y_{i'j}) \\
&= -\frac{1}{T}\sum_{j=1}^{T}\sum_{i=1}^{T}\left[\sigma_\epsilon^2 + \frac{1}{T}\sum_{t=1}^{T}\{X_{ij}(t) - \bar{X}_{ij}(\cdot)\}^2\right] \\
&\quad + \frac{1}{T^2(T-1)}\sum_{j=1}^{T}\sum_{i\neq i'}\left\{\sum_{t=1}^{T}X_{ij}(t)X_{i'j}(t) - T\bar{X}_{ij}(\cdot)\bar{X}_{i'j}(\cdot)\right\},
\end{aligned}
$$

and

$$
\sum_{j=1}^{T}\sum_{i\neq i'}\left\{\sum_{t=1}^{T}X_{ij}(t)X_{i'j}(t)\right\} = \sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{t=1}^{T}X_{ij}(t)\{T\bar{X}_{\cdot j}(t) - X_{ij}(t)\},
$$

$$
\sum_{j=1}^{T}\sum_{i\neq i'}\{-T\bar{X}_{ij}(\cdot)\bar{X}_{i'j}(\cdot)\} = -\sum_{i=1}^{T}\sum_{j=1}^{T}T\bar{X}_{ij}(\cdot)\{T\bar{X}_{\cdot j}(\cdot) - \bar{X}_{ij}(\cdot)\}.
$$

By combining all these terms, we have that (A.2) equals

$$
\begin{aligned}
&-2(T-1)\sigma_\epsilon^2 - \frac{2}{T(T-1)}\sum_{i=1}^{T}\sum_{j=1}^{T}\sum_{t=1}^{T}\{X_{ij}(t) - \bar{X}_{ij}(\cdot)\}^2 + \frac{1}{T-1}\sum_{i=1}^{T}\sum_{t=1}^{T}\{\bar{X}_{i\cdot}(t) - \bar{X}_{i\cdot}(\cdot)\}^2 \\
&\quad + \frac{1}{T-1}\sum_{j=1}^{T}\sum_{t=1}^{T}\{\bar{X}_{\cdot j}(t) - \bar{X}_{\cdot j}(\cdot)\}^2 + \frac{2}{T-1}\sum_{t=1}^{T}\sigma_\eta^2(t) \\
&\quad - \frac{2}{(T-1)^2}\sum_{t\neq t'}r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}.
\end{aligned}
$$

We rewrite this expression to obtain

$$-2(T-1)\sigma_\epsilon^2 + \frac{2}{T-1}\sum_{t=1}^T \sigma_\eta^2(t) - \frac{2}{(T-1)^2}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}$$
$$-\frac{1}{T-1}\left[\sum_{i=1}^T\sum_{t=1}^T\{R_i(t)-\bar{R}_i(\cdot)\}^2 + \sum_{j=1}^T\sum_{t=1}^T\{C_j(t)-\bar{C}_j(\cdot)\}^2 + \frac{2}{T}\sum_{i=1}^T\sum_{j=1}^T\sum_{t=1}^T\{\eta_{ij}(t)-\bar{\eta}_{ij}(\cdot)\}^2\right].$$

To finish with the third term, we note that

$$\bar{X}_{ij}(\cdot) = \bar{X}_{i\cdot}(\cdot) + \bar{X}_{\cdot j}(\cdot) + \bar{\eta}_{ij}(\cdot) - \bar{X}_{\cdot\cdot}(\cdot),$$

so that

$$\sum_{i=1}^T\sum_{j=1}^T\{\bar{X}_{ij}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2 = T\sum_{i=1}^T\{\bar{X}_{i\cdot}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2 + T\sum_{j=1}^T\{\bar{X}_{\cdot j}(\cdot)-\bar{X}_{\cdot\cdot}(\cdot)\}^2 + \sum_{i=1}^T\sum_{j=1}^T\bar{\eta}_{ij}(\cdot)^2.$$

Hence, we write (A.3) as

$$\sum_{t=1}^T\sigma_\eta^2(t) + \sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')} - T\sum_{t=1}^T\{\bar{X}_{\cdot\cdot}(t)-\bar{X}_{\cdot\cdot}(\cdot)\}^2.$$

We add all these three terms to obtain (after algebraic simplification)

$$\begin{aligned}\mathbb{E}\{(T-1)(T-2)S_0^2\} &= (T-1)(T-2)\sigma_\epsilon^2 + \frac{(T-2)^2}{T-1}\sum_{t=1}^T\sigma_\eta^2(t)\\ &\quad+\frac{2(T-2)}{(T-1)^2}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}\\ &\quad+\frac{T-2}{T-1}\left[\sum_{i=1}^T\sum_{t=1}^T\{R_i(t)-\bar{R}_i(\cdot)\}^2+\sum_{j=1}^T\sum_{t=1}^T\{C_j(t)-\bar{C}_j(\cdot)\}^2\right].\end{aligned}$$

From before, we have

$$\begin{aligned}\mathbb{E}\{(T-1)S_1^2\} &= T\sum_{t=1}^T \mathrm{Var}\{\bar{x}^o_{\cdot\cdot}(t)\} - T^2\mathrm{Var}(\bar{y}_{\cdot\cdot}) + T\sum_{t=1}^T\{\bar{X}_{\cdot\cdot}(t)-\bar{X}_{\cdot\cdot}(\cdot)\}^2\\ &= (T-1)\sigma_\epsilon^2 + \sum_{t=1}^T\sigma_\eta^2(t) + \frac{1}{(T-1)^2}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}\\ &\quad+T\sum_{t=1}^T\{\bar{X}_{\cdot\cdot}(t)-\bar{X}_{\cdot\cdot}(\cdot)\}^2.\end{aligned}$$

Thus, for LSs, the expected mean residual sum of squares is

$$\begin{aligned}\mathbb{E}(S_0^2) &= \sigma_\epsilon^2 + \frac{T-2}{(T-1)^2}\sum_{t=1}^T\sigma_\eta^2(t) + \frac{2}{(T-1)^3}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}\\ &\quad+\frac{1}{(T-1)^2}\left[\sum_{i=1}^T\sum_{t=1}^T\{R_i(t)-\bar{R}_i(\cdot)\}^2+\sum_{j=1}^T\sum_{t=1}^T\{C_j(t)-\bar{C}_j(\cdot)\}^2\right],\end{aligned}$$

and the expected mean treatment sum of squares is

$$\mathbb{E}(S_1^2) = \sigma_\epsilon^2 + \frac{1}{T-1}\sum_{t=1}^{T}\sigma_\eta^2(t) + \frac{1}{(T-1)^3}\sum_{t\neq t'} r(t,t')\sqrt{\sigma_\eta^2(t)\sigma_\eta^2(t')}$$
$$+ \frac{T}{T-1}\sum_{t=1}^{T}\{\bar{X}_{\cdot\cdot}(t) - \bar{X}_{\cdot\cdot}(\cdot)\}^2.$$

A.3 Latin Square Probabilities

For the LS assignment mechanism, treatment labels are fixed and we randomly choose a LS of order T, with $T \in \mathbb{Z}_{\geq 3}$ a fixed integer.

LEMMA A.1. *For any cell (i, j) and treatment t, there exists at least one LS with treatment t in (i, j).*

PROOF. The Cayley table of the cyclic group $(\mathbb{Z}/T\mathbb{Z}, +)$ is a LS. Because treatment t appears in row i, switch two columns so that t is in cell (i, j). The transformed square is a LS. □

LEMMA A.2. *The number of LSs with t' in cell (i, j) equals the number of LSs with t in cell (i, j), where $t \neq t'$.*

PROOF. Consider two distinct LSs with treatment t in cell (i, j). In the interior of each square, relabel all the t cells as t' and all the t' cells as t. The transformed squares remain distinct LSs. Hence the number of LSs with t' in cell (i, j) is greater than or equal to the number of LSs with t in (i, j), and so by symmetry must be equal. □

COROLLARY A.3. *For $t \neq t'$, $\Pr\{W_{ij}(t) = 1\} = \Pr\{W_{ij}(t') = 1\}$.*

PROPOSITION A.4. *For any cell (i, j) and treatment t, $\Pr\{W_{ij}(t) = 1\} = 1/T$.*

PROOF. From the definition of a LS,

$$1 = \sum_{t=1}^{T}\Pr\{W_{ij}(t) = 1\} = T\Pr\{W_{ij}(1) = 1\}$$
$$\Rightarrow \Pr\{W_{ij}(t) = 1\} = \frac{1}{T}\ \forall t \in \{1, \ldots, T\}.$$

□

We now calculate probabilities for distinct cells. From the definition of a LS,

$$\Pr\{W_{ij}(t) = W_{ij'}(t) = 1\} = \Pr\{W_{ij}(t) = W_{i'j}(t) = 1\} = 0$$

for $i \neq i'$, $j \neq j'$. First are probabilities for cells in the same row/column with different treatments.

LEMMA A.5. *The number of LSs with t in (i, j) and t' in (i, j') equals the number of LSs with t in (i, j) and t'' in (i, j'), where $t, t', t'' \in \{1, \ldots, T\}$ are all distinct and $j \neq j'$.*

PROOF. For any two distinct LSs with t in (i,j) and t' in (i,j'), relabeling all the t' as t'' and all the t'' as t' in their interiors yields two distinct LSs with t in (i,j) and t'' in (i,j'). This lemma follows by symmetry. $\square$

LEMMA A.6. *The number of LSs with t in (i,j) and t' in (i,j') equals the number of LSs with t' in (i,j) and t in (i,j'), where $t \neq t', j \neq j'$.*

PROOF. For any two distinct LSs with t in (i,j) and t' in (i,j'), relabeling all the t' as t and all the t as t' in their interiors yields two distinct LSs with t' in (i,j) and t in (i,j'). This lemma follows by symmetry. $\square$

COROLLARY A.7. *For $j \neq j'$, $\Pr\{W_{ij}(t) = W_{ij'}(t') = 1\}$ is constant as a function of (distinct) $t, t' \in \{1, \dots, T\}$.*

PROPOSITION A.8. *For $j \neq j', t \neq t'$, $\Pr\{W_{ij}(t) = W_{ij'}(t') = 1\} = 1/\{T(T-1)\}$.*

PROOF. From the definition of a LS, the probability of two different treatments being assigned to (i,j) and (i,j') is equal to 1. Hence

$$1 = \sum_{t=1}^{T} \sum_{t \neq t'} \Pr\{W_{ij}(t) = W_{ij'}(t') = 1\} = T(T-1)\Pr\{W_{ij}(1) = W_{ij'}(2) = 1\}$$

$$\Rightarrow \Pr\{W_{ij}(t) = W_{ij'}(t') = 1\} = \frac{1}{T(T-1)} \ \forall t \neq t'.$$

$\square$

By symmetry, we obtain the following.

PROPOSITION A.9. *For $i \neq i', t \neq t'$, $\Pr\{W_{ij}(t) = W_{i'j}(t') = 1\} = 1/\{T(T-1)\}$.*

We now consider different rows and columns with the same treatments.

LEMMA A.10. *For distinct cells $(i_1, j_1), \dots, (i_T, j_T)$, with $i_1, \dots, i_T \in \{1, \dots, T\}$ all distinct and $j_1, \dots, j_T \in \{1, \dots, T\}$ all distinct, there exists at least one LS with treatment t in all these cells.*

PROOF. The Cayley table of the cyclic group $(\mathbb{Z}/T\mathbb{Z}, +)$ is a LS. For each row in this LS, switch two columns to ensure that t is in all the cells $(i_1, j_1), \dots, (i_T, j_T)$, which can be done as each treatment occurs only once in any row and column. $\square$

LEMMA A.11. *The number of LSs with t in all of $(i_1, j_1), \dots, (i_T, j_T)$ equals the number of LSs with t in all of $(i'_1, j'_1), \dots, (i'_T, j'_T)$, where $i_1, \dots, i_T$ are distinct, $j_1, \dots, j_T$ are distinct, and similarly $i'_1, \dots, i'_T$ are distinct, $j'_1, \dots, j'_T$ are distinct.*

PROOF. For any two distinct LSs with t in all of $(i_1, j_1), \dots, (i_T, j_T)$, simply switch the required columns in the desired order to obtain two distinct LSs with t in all of $(i'_1, j'_1), \dots, (i'_T, j'_T)$. This lemma then follows by symmetry. $\square$

Corollary A.12. *For any T cells $(i_1, j_1), \ldots, (i_T, j_T)$, with $i_1, \ldots, i_T$ all distinct and $j_1, \ldots, j_T$ all distinct,* $\Pr\{W_{i_1j_1}(t) = \ldots = W_{i_Tj_T}(t) = 1\} = 1/T!$.

Proof. From the definition of a LS, the probability that treatment t is in T distinct cells is equal to 1. Taking into account the $T!$ possible permutations of the columns of distinct cells and the results above,

$$T! \times \Pr\{W_{i_1j_1}(t) = \ldots = W_{i_Tj_T}(t) = 1\} = 1$$
$$\Rightarrow \Pr\{W_{i_1j_1}(t) = \ldots = W_{i_Tj_T}(t) = 1\} = \frac{1}{T!}.$$

□

Proposition A.13. *For $i_1 \neq i_2, j_1 \neq j_2$,* $\Pr\{W_{i_1j_1}(t) = W_{i_2j_2}(t) = 1\} = 1/\{T(T-1)\}$.

Proof.

$$\begin{aligned}\Pr\{W_{i_1j_1}(t) = W_{i_2j_2}(t) = 1\} &= \sum_{(i_3,j_3),\ldots,(i_T,j_T)} \Pr\{W_{i_1j_1}(t) = \ldots = W_{i_Tj_T}(t) = 1\} \\ &= \frac{(T-2)!}{T!}\end{aligned}$$

□

We finally consider different rows and columns with different treatments.

Lemma A.14. *The number of LSs with t in (i, j) and t' in (i', j') equals the number of LSs with t in (i, j) and t'' in (i', j'), and equals the number of LSs with t' in (i, j) and t in (i', j'), where $i \neq i', j \neq j'$, and t, t', t'' are distinct.*

Proof. This follows by the same reasoning as before. □

Corollary A.15. *For $i \neq i', j \neq j'$, and distinct t, t', t'',* $\Pr\{W_{ij}(t) = W_{i'j'}(t') = 1\} = \Pr\{W_{ij}(t) = W_{i'j'}(t'') = 1\} = \Pr\{W_{ij}(t') = W_{i'j'}(t) = 1\}$.

Proposition A.16. *For $i \neq i', j \neq j', t \neq t'$,* $\Pr\{W_{ij}(t) = W_{i'j'}(t') = 1\} = (T-2)/\{T(T-1)^2\}$.

Proof. From the definition of a LS, and our previous results,

$$\begin{aligned}1 &= \Pr[W_{ij}(t) = W_{i'j'}(t') = 1 \text{ for some } t, t' \in \{1, \ldots, T\}] \\ &= \sum_{t=1}^{T}\sum_{t'=1}^{T} \Pr\{W_{ij}(t) = W_{i'j'}(t') = 1\} \\ &= \frac{T}{T(T-1)} + T(T-1)\Pr\{W_{ij}(1) = W_{i'j'}(2) = 1\} \\ \Rightarrow &\Pr\{W_{ij}(t) = W_{i'j'}(t') = 1\} = \frac{T-2}{T(T-1)^2} \;\forall t \neq t'.\end{aligned}$$

□

REFERENCES

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