# Proofs and Selected Design Augmentations Supplement to "An Integrative Framework for Geometric and Hidden Projections in Three-Level Fractional Factorial Designs"

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## 1. Proofs

*Proof of Lemma 1.* We have $|\mathcal{F}| = \sum_{x \in \mathcal{D}_s} F_{\mathcal{F}}(x) = \sum_{x \in \mathcal{D}_s} \sum_{I \in \mathcal{I}} \sum_{T \in \mathcal{T}_I} b_{\mathcal{F},I,T} X_{I,T}(x) = 3^s b_{\mathcal{F},\phi,\phi}$, and so $b_{\mathcal{F},\phi,\phi} = 3^{-s}|\mathcal{F}|$. □

*Proof of Lemma 2.* For one factor $A_{j_1}$, as $b_{\mathcal{H},\phi,\phi} = 3^{-p}(|\mathcal{F}|+|\mathcal{G}|) = 3^{s-p} b_{\mathcal{F},\phi,\phi} + b_{\mathcal{G},\phi,\phi}$,

$$\begin{aligned}\begin{pmatrix} b_{\mathcal{H},j_1,\mathrm{L}} \\ b_{\mathcal{H},j_1,\mathrm{Q}} \end{pmatrix} &= 2^{-1}3^{1-p} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} |\mathcal{F}_{j_1,\mathrm{L}}| + |\mathcal{G}_{j_1,\mathrm{L}}| \\ |\mathcal{F}_{j_1,\mathrm{Q}}| + |\mathcal{G}_{j_1,\mathrm{Q}}| \end{pmatrix} - \begin{pmatrix} 0 \\ 3^{s-p} b_{\mathcal{F},\phi,\phi} + b_{\mathcal{G},\phi,\phi} \end{pmatrix} \\ &= 3^{s-p} \begin{pmatrix} b_{\mathcal{F},j_1,\mathrm{L}} \\ b_{\mathcal{F},j_1,\mathrm{Q}} \end{pmatrix} + \begin{pmatrix} b_{\mathcal{G},j_1,\mathrm{L}} \\ b_{\mathcal{G},j_1,\mathrm{Q}} \end{pmatrix}.\end{aligned}$$

For distinct $A_{j_1}$ and $A_{j_2}$, $B_{\mathcal{H},j_1j_2,\mathrm{LQ}} = 3^{s-p} b_{\mathcal{F},j_1,\mathrm{L}} + b_{\mathcal{G},j_1,\mathrm{L}}$, $B_{\mathcal{H},j_1j_2,\mathrm{QL}} = 3^{s-p} b_{\mathcal{F},j_2,\mathrm{L}} + b_{\mathcal{G},j_2,\mathrm{L}}$, and $B_{\mathcal{H},j_1j_2,\mathrm{QQ}} = 3^{s-p}(b_{\mathcal{F},\phi,\phi} + b_{\mathcal{F},j_1,\mathrm{Q}} + b_{\mathcal{F},j_2,\mathrm{Q}}) + b_{\mathcal{G},\phi,\phi} + b_{\mathcal{G},j_1,\mathrm{Q}} + b_{\mathcal{G},j_2,\mathrm{Q}}$. As $|\mathcal{H}_{j_1j_2,TS}| = |\mathcal{F}_{j_1j_2,TS}| + |\mathcal{G}_{j_1j_2,TS}|$ for all $T,S \in \{\mathrm{L},\mathrm{Q}\}$, we have that $(A_{j_1} \otimes A_{j_2})_{\mathcal{H}} = 3^{s-p}(A_{j_1} \otimes A_{j_2})_{\mathcal{F}} + (A_{j_1} \otimes A_{j_2})_{\mathcal{G}}$. The general result then follows by induction. □

*Proof of Theorem 1.* By Lemma 1, the construction of the smallest follow-up design is equivalent to identifying a valid set of indicator function coefficients $\mathbf{b}_{\mathcal{G}}$ with minimum $b_{\mathcal{G},\phi,\phi}$. In order for $\mathbf{b}_{\mathcal{G}}$ to be valid, each row of $\mathbf{M}_p\mathbf{b}_{\mathcal{G}}$ must be either 0 or 1,

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which holds if and only if $(\mathbf{M}_p\mathbf{b}_{\mathcal{G}}) \odot (\mathbf{1}_{|\mathcal{D}_p|} - \mathbf{M}_p\mathbf{b}_{\mathcal{G}}) = (0, \ldots, 0)^{\mathsf{T}}$. Lemma 2 specifies the indicator function coefficients for $\mathcal{G}$ that are necessary to yield the desired aliasing structure for $\mathcal{H}$. Thus, the construction of the smallest follow-up design is equivalent to finding $\mathbf{b}_{\mathcal{G}}$ that minimizes $b_{\mathcal{G},\phi,\phi}$ subject to these constraints. □

*Proof of Corollary 1.* By inspection of the second-order model, an augmented design $\mathcal{H}$ satisfies the second-order orthogonality criterion if and only if all of its indicator function coefficients of the form $b_{\mathcal{H},i,\mathrm{L}}$, $b_{\mathcal{H},i,\mathrm{Q}}$, $b_{\mathcal{H},ij,\mathrm{LL}}$, $b_{\mathcal{H},ij,\mathrm{LQ}}$, $b_{\mathcal{H},ij,\mathrm{QQ}}$, and $b_{\mathcal{H},ijk,\mathrm{LLL}}$ are zero. We enter these conditions into Theorem 1 to obtain the optimization. □

*Proof of Lemma 3.* For $(i,j) \in \{1, \ldots, (p+1)(p+2)/2\}$, entry $(i,j)$ of $\mathbf{X}_{\mathcal{F}}^{\mathsf{T}}\mathbf{X}_{\mathcal{F}}$ is the dot product of columns $i$ and $j$ of $\mathbf{X}_{\mathcal{F}}$, which is equivalently expressed as

$$\sum_{x \in \mathcal{F}} Z_{(i)}(x) Z_{(j)}(x) = \sum_{x \in \mathcal{D}_p} F_{\mathcal{F}}(x) Z_{(i)}(x) Z_{(j)}(x).$$

□

*Proof of Theorem 2.* The entries of $\mathbf{X}_{\mathcal{H}}^{\mathsf{T}}\mathbf{X}_{\mathcal{H}}$ are derived by combining Lemma 3 with the aliasing relations induced by the second-order orthogonality criterion to calculate the inner products of contrast vectors across the columns of $\mathbf{X}_{\mathcal{H}}$. Then $\det\left(\mathbf{X}_{\mathcal{H}}^{\mathsf{T}}\mathbf{X}_{\mathcal{H}}\right)$ and $\left(\mathbf{X}_{\mathcal{H}}^{\mathsf{T}}\mathbf{X}_{\mathcal{H}}\right)^{-1}$ follow from the block-matrix structure for $\mathbf{X}_{\mathcal{H}}^{\mathsf{T}}\mathbf{X}_{\mathcal{H}}$. □

## 2. Selected Designs and Augmentations

Selected three-level fractional factorial designs, their augmentations, and further details for the case studies considered in Sections 1, 3.2, 4.2, and 4.3 are summarized in the following tables and figures.

Table 1: The DSD(13), with $A_1, \ldots, A_6$ denoting its six factors and $-1, 0$, and 1 their levels (Jones and Nachtsheim, 2011, p. 5).

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|
| 1 | 0 | 1 | -1 | -1 | -1 | -1 |
| 2 | 0 | -1 | 1 | 1 | 1 | 1 |
| 3 | 1 | 0 | -1 | 1 | 1 | -1 |
| 4 | -1 | 0 | 1 | -1 | -1 | 1 |
| 5 | -1 | -1 | 0 | 1 | -1 | -1 |
| 6 | 1 | 1 | 0 | -1 | 1 | 1 |
| 7 | -1 | 1 | 1 | 0 | 1 | -1 |
| 8 | 1 | -1 | -1 | 0 | -1 | 1 |
| 9 | 1 | -1 | 1 | -1 | 0 | -1 |
| 10 | -1 | 1 | -1 | 1 | 0 | 1 |
| 11 | 1 | 1 | 1 | 1 | -1 | 0 |
| 12 | -1 | -1 | -1 | -1 | 1 | 0 |
| 13 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2: The $3^{3-1}_{\text{III}}$ regular fractional factorial design. Under the orthogonal components system, in which our levels of $-1, 0$, and 1 are coded as $0, 1$, and 2 modulo 3, the third factor is defined as the sum of the first two factors modulo 3.

| Run | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| 1 | -1 | -1 | -1 |
| 2 | -1 | 0 | 0 |
| 3 | -1 | 1 | 1 |
| 4 | 0 | -1 | 0 |
| 5 | 0 | 0 | 1 |
| 6 | 0 | 1 | -1 |
| 7 | 1 | -1 | 1 |
| 8 | 1 | 0 | -1 |
| 9 | 1 | 1 | 0 |

Table 3: A follow-up design to the $3^{3-1}_{\mathrm{III}}$ regular fraction in Table 2 that yields an augmentation satisfying the second-order orthogonality criterion for all factors.

| Run | $A_1$ | $A_2$ | $A_3$ |
|---|---|---|---|
| 10 | -1 | -1 | 1 |
| 11 | -1 | 0 | 0 |
| 12 | -1 | 1 | -1 |
| 13 | 0 | -1 | 0 |
| 14 | 0 | 0 | -1 |
| 15 | 0 | 1 | 1 |
| 16 | 1 | -1 | -1 |
| 17 | 1 | 0 | 1 |
| 18 | 1 | 1 | 0 |

Table 4: The DSD(11) with five factors (Jones and Nachtsheim, 2011, p. 5).

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ |
|---|---|---|---|---|---|
| 1 | 0 | 1 | 1 | -1 | -1 |
| 2 | 0 | -1 | -1 | 1 | 1 |
| 3 | 1 | 0 | -1 | -1 | 1 |
| 4 | -1 | 0 | 1 | 1 | -1 |
| 5 | 1 | -1 | 0 | 1 | -1 |
| 6 | -1 | 1 | 0 | -1 | 1 |
| 7 | 1 | -1 | 1 | 0 | 1 |
| 8 | -1 | 1 | -1 | 0 | -1 |
| 9 | 1 | 1 | 1 | 1 | 0 |
| 10 | -1 | -1 | -1 | -1 | 0 |
| 11 | 0 | 0 | 0 | 0 | 0 |

Table 5: Two follow-up designs to the projection onto factors $A_1, A_2, A_3$, and $A_4$ of the DSD(11) in Table 4 that yield augmented designs satisfying the second-order orthogonality criterion. The right set of runs yields an augmented design that is A- and D-optimum with respect to the second-order model among all candidate augmentations.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 12 | -1 | -1 | 0 | 0 | 12 | -1 | -1 | 0 | 0 |
| 13 | -1 | -1 | 1 | -1 | 13 | -1 | -1 | 1 | -1 |
| 14 | -1 | 0 | 0 | 1 | 14 | -1 | 0 | -1 | 1 |
| 15 | -1 | 0 | 1 | 1 | 15 | -1 | 0 | 1 | 0 |
| 16 | -1 | 1 | -1 | 0 | 16 | -1 | 1 | 0 | 1 |
| 17 | 0 | -1 | -1 | 1 | 17 | 0 | -1 | -1 | 0 |
| 18 | 0 | -1 | 0 | -1 | 18 | 0 | -1 | 1 | 1 |
| 19 | 0 | 0 | -1 | 0 | 19 | 0 | 0 | 0 | -1 |
| 20 | 0 | 0 | 1 | 0 | 20 | 0 | 0 | 0 | 1 |
| 21 | 0 | 1 | 0 | 1 | 21 | 0 | 1 | -1 | -1 |
| 22 | 0 | 1 | 1 | -1 | 22 | 0 | 1 | 1 | 0 |
| 23 | 1 | -1 | 1 | 0 | 23 | 1 | -1 | 0 | -1 |
| 24 | 1 | 0 | -1 | -1 | 24 | 1 | 0 | -1 | 0 |
| 25 | 1 | 0 | 0 | -1 | 25 | 1 | 0 | 1 | -1 |
| 26 | 1 | 1 | -1 | 1 | 26 | 1 | 1 | -1 | 1 |
| 27 | 1 | 1 | 0 | 0 | 27 | 1 | 1 | 0 | 0 |

Table 6: The second follow-up design of size 16 to the first four factors of the DSD(11) that yields an augmented design that is A- and D-optimum with respect to the second-order model and satisfies the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| 12 | -1 | -1 | 0 | 1 |
| 13 | -1 | -1 | 1 | -1 |
| 14 | -1 | 0 | -1 | 1 |
| 15 | -1 | 0 | 0 | 0 |
| 16 | -1 | 1 | 1 | 0 |
| 17 | 0 | -1 | 0 | -1 |
| 18 | 0 | -1 | 1 | 0 |
| 19 | 0 | 0 | -1 | -1 |
| 20 | 0 | 0 | 1 | 1 |
| 21 | 0 | 1 | -1 | 0 |
| 22 | 0 | 1 | 0 | 1 |
| 23 | 1 | -1 | -1 | 0 |
| 24 | 1 | 0 | 0 | 0 |
| 25 | 1 | 0 | 1 | -1 |
| 26 | 1 | 1 | -1 | 1 |
| 27 | 1 | 1 | 0 | -1 |

Table 7: The follow-up design of size 16 to the first four factors of the DSD(11) that yields an augmented design that is E- and G-optimum with respect to the second-order model and satisfies the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| 12 | -1 | -1 | 0 | 0 |
| 13 | -1 | -1 | 1 | -1 |
| 14 | -1 | 0 | -1 | 1 |
| 15 | -1 | 0 | 0 | 1 |
| 16 | -1 | 1 | 1 | 0 |
| 17 | 0 | -1 | 0 | -1 |
| 18 | 0 | -1 | 1 | 1 |
| 19 | 0 | 0 | -1 | 0 |
| 20 | 0 | 0 | 1 | 0 |
| 21 | 0 | 1 | -1 | -1 |
| 22 | 0 | 1 | 0 | 1 |
| 23 | 1 | -1 | -1 | 0 |
| 24 | 1 | 0 | 0 | -1 |
| 25 | 1 | 0 | 1 | -1 |
| 26 | 1 | 1 | -1 | 1 |
| 27 | 1 | 1 | 0 | 0 |

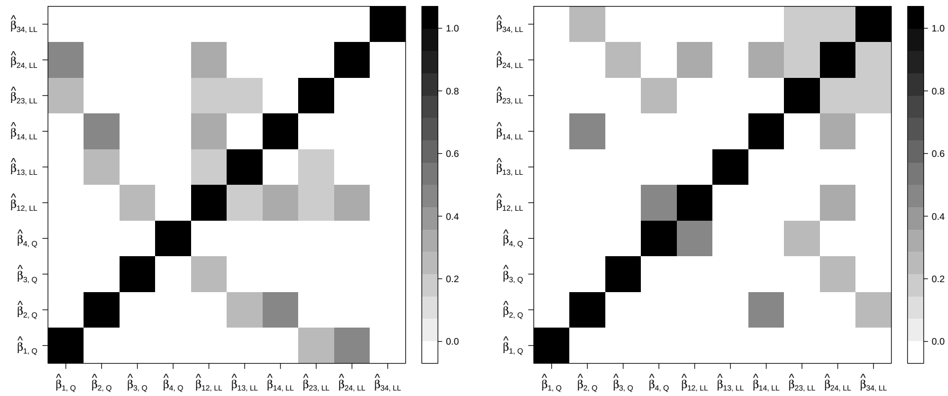

Figure 1: Absolute correlations among second-order contrasts for the A- and D-optimum augmented designs for factors $A_1, A_2, A_3$, and $A_4$ of the DSD(11) that are formed from the corresponding follow-up runs in Tables 5 and 6. The absolute correlations are in grayscale, with white denoting zero correlation and black an absolute correlation of one. Under the second-order orthogonality criterion, all cells involving at least one linear main effect (excluding diagonals) or two distinct quadratic main effects are white.

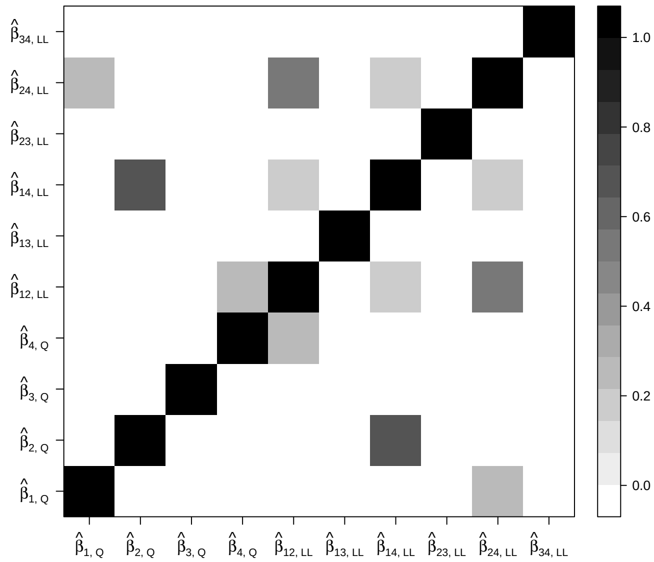


Figure 2: Absolute correlations among second-order contrasts for the E- and G-optimum augmented design for factors $A_1, A_2, A_3$, and $A_4$ of the DSD(11) that is formed from the corresponding follow-up runs in Table 7.

Table 8: Two sets of follow-up runs to the projection onto factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) in Table 1 that yield augmented designs satisfying the second-order orthogonality criterion. The right set of runs yields an augmented design that is A-, D-, and E-optimum with respect to the second-order model among all candidate augmentations.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 14 | 1 | 1 | 1 | 0 | 14 | -1 | -1 | 1 | 0 |
| 15 | 1 | 0 | 0 | 1 | 15 | -1 | 0 | -1 | 0 |
| 16 | 1 | 0 | -1 | 0 | 16 | -1 | 0 | 0 | 1 |
| 17 | 1 | -1 | 0 | -1 | 17 | -1 | 1 | 0 | -1 |
| 18 | 0 | 1 | 0 | 0 | 18 | 0 | -1 | -1 | -1 |
| 19 | 0 | 1 | -1 | -1 | 19 | 0 | -1 | 0 | 0 |
| 20 | 0 | 0 | 1 | -1 | 20 | 0 | 0 | -1 | 1 |
| 21 | 0 | 0 | -1 | 1 | 21 | 0 | 0 | 1 | -1 |
| 22 | 0 | -1 | 1 | 1 | 22 | 0 | 1 | 0 | 0 |
| 23 | 0 | -1 | 0 | 0 | 23 | 0 | 1 | 1 | 1 |
| 24 | -1 | 1 | 0 | 1 | 24 | 1 | -1 | 0 | 1 |
| 25 | -1 | 0 | 1 | 0 | 25 | 1 | 0 | 0 | -1 |
| 26 | -1 | 0 | 0 | -1 | 26 | 1 | 0 | 1 | 0 |
| 27 | -1 | -1 | -1 | 0 | 27 | 1 | 1 | -1 | 0 |

Table 9: Two sets of follow-up runs to the projection onto factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) in Table 1 that yield augmented designs satisfying the second-order orthogonality criterion and are A- and E-optimum (left), and G-optimum (right), respectively, with respect to the second-order model among all candidate augmentations.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| 14 | -1 | -1 | 0 | 0 |
| 15 | -1 | 0 | -1 | 0 |
| 16 | -1 | 0 | 1 | 1 |
| 17 | -1 | 1 | 0 | -1 |
| 18 | 0 | -1 | -1 | 0 |
| 19 | 0 | -1 | 1 | -1 |
| 20 | 0 | 0 | 0 | -1 |
| 21 | 0 | 0 | 0 | 1 |
| 22 | 0 | 1 | -1 | 1 |
| 23 | 0 | 1 | 1 | 0 |
| 24 | 1 | -1 | 0 | 1 |
| 25 | 1 | 0 | -1 | -1 |
| 26 | 1 | 0 | 1 | 0 |
| 27 | 1 | 1 | 0 | 0 |

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| 14 | -1 | -1 | 0 | 0 |
| 15 | -1 | 0 | 0 | 1 |
| 16 | -1 | 0 | 1 | 0 |
| 17 | -1 | 1 | -1 | -1 |
| 18 | 0 | -1 | -1 | 0 |
| 19 | 0 | -1 | 0 | -1 |
| 20 | 0 | 0 | -1 | 1 |
| 21 | 0 | 0 | 1 | -1 |
| 22 | 0 | 1 | 0 | 1 |
| 23 | 0 | 1 | 1 | 0 |
| 24 | 1 | -1 | 1 | 1 |
| 25 | 1 | 0 | -1 | 0 |
| 26 | 1 | 0 | 0 | -1 |
| 27 | 1 | 1 | 0 | 0 |

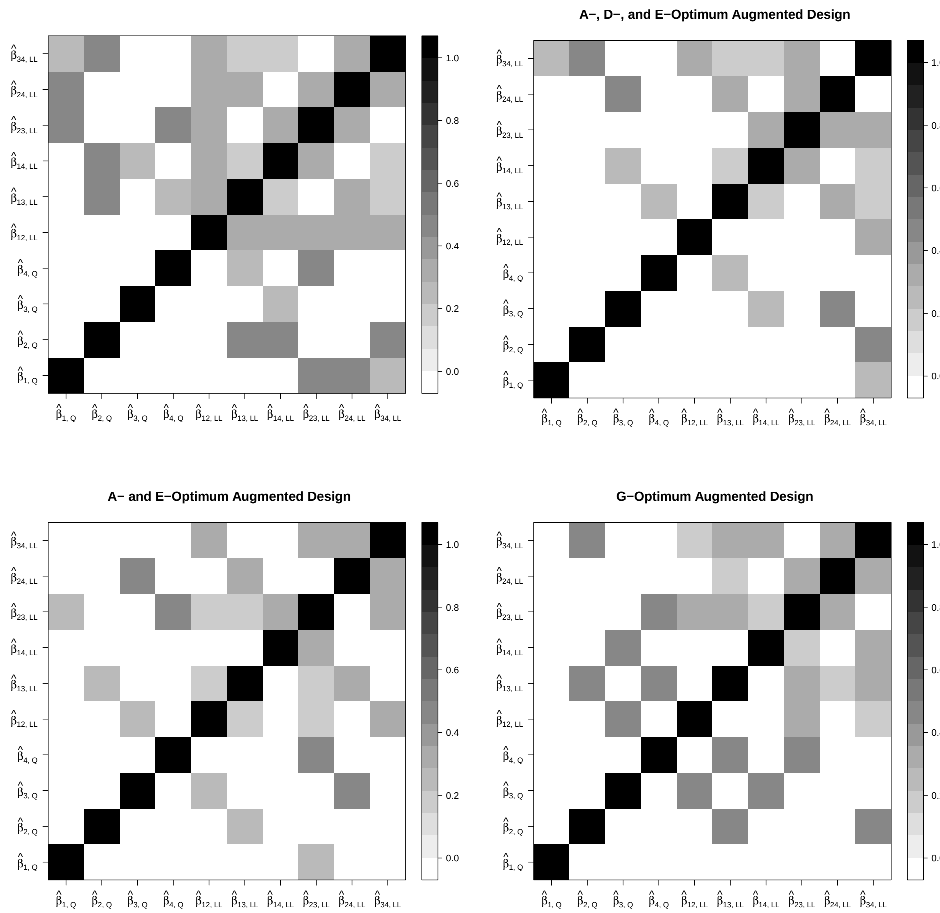


Figure 3: Absolute correlations of second-order contrasts for distinct candidate augmented designs for the first four factors of the DSD(13) that correspond to the follow-up runs in Tables 8 and 9. One candidate design is optimum with respect to the A-, D-, and E-optimality criteria, another is optimum only with respect to the A- and E-optimality criteria, and the last is optimum only with respect to the G-optimality criteria, for the second-order model.

Table 10: The first two follow-up designs of size 14 to the factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) that yield augmented designs that are A-, D-, and E-optimum with respect to the second-order model and satisfy the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| 14 | -1 | -1 | 1 | 0 |
| 15 | -1 | 0 | -1 | 0 |
| 16 | -1 | 0 | 0 | 1 |
| 17 | -1 | 1 | 0 | -1 |
| 18 | 0 | -1 | -1 | -1 |
| 19 | 0 | -1 | 0 | 0 |
| 20 | 0 | 0 | -1 | 1 |
| 21 | 0 | 0 | 1 | -1 |
| 22 | 0 | 1 | 0 | 0 |
| 23 | 0 | 1 | 1 | 1 |
| 24 | 1 | -1 | 0 | 1 |
| 25 | 1 | 0 | 0 | -1 |
| 26 | 1 | 0 | 1 | 0 |
| 27 | 1 | 1 | -1 | 0 |

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|
| 14 | -1 | -1 | 0 | 0 |
| 15 | -1 | 0 | 0 | -1 |
| 16 | -1 | 0 | 1 | 1 |
| 17 | -1 | 1 | -1 | 0 |
| 18 | 0 | -1 | -1 | 1 |
| 19 | 0 | -1 | 0 | -1 |
| 20 | 0 | 0 | -1 | 0 |
| 21 | 0 | 0 | 1 | 0 |
| 22 | 0 | 1 | 0 | 1 |
| 23 | 0 | 1 | 1 | -1 |
| 24 | 1 | -1 | 1 | 0 |
| 25 | 1 | 0 | -1 | -1 |
| 26 | 1 | 0 | 0 | 1 |
| 27 | 1 | 1 | 0 | 0 |

Table 11: The second two follow-up designs of size 14 to the factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) that yield augmented designs that are A-, D-, and E-optimum with respect to the second-order model and satisfy the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 14 | -1 | -1 | 1 | 0 | 14 | -1 | -1 | -1 | 0 |
| 15 | -1 | 0 | -1 | -1 | 15 | -1 | 0 | 0 | 0 |
| 16 | -1 | 0 | 0 | 1 | 16 | -1 | 0 | 1 | 1 |
| 17 | -1 | 1 | 0 | 0 | 17 | -1 | 1 | 0 | -1 |
| 18 | 0 | -1 | -1 | 1 | 18 | 0 | -1 | 0 | -1 |
| 19 | 0 | -1 | 0 | -1 | 19 | 0 | -1 | 1 | 0 |
| 20 | 0 | 0 | -1 | 0 | 20 | 0 | 0 | -1 | 1 |
| 21 | 0 | 0 | 1 | 0 | 21 | 0 | 0 | 1 | -1 |
| 22 | 0 | 1 | 0 | 1 | 22 | 0 | 1 | -1 | 0 |
| 23 | 0 | 1 | 1 | -1 | 23 | 0 | 1 | 0 | 1 |
| 24 | 1 | -1 | 0 | 0 | 24 | 1 | -1 | 0 | 1 |
| 25 | 1 | 0 | 0 | -1 | 25 | 1 | 0 | -1 | -1 |
| 26 | 1 | 0 | 1 | 1 | 26 | 1 | 0 | 0 | 0 |
| 27 | 1 | 1 | -1 | 0 | 27 | 1 | 1 | 1 | 0 |

Table 12: The final two follow-up designs of size 14 to the factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) that yield augmented designs that are A-, D-, and E-optimum with respect to the second-order model and satisfy the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 14 | -1 | -1 | 0 | -1 | 14 | -1 | -1 | 1 | 0 |
| 15 | -1 | 0 | -1 | 0 | 15 | -1 | 0 | -1 | 1 |
| 16 | -1 | 0 | 1 | 1 | 16 | -1 | 0 | 0 | 0 |
| 17 | -1 | 1 | 0 | 0 | 17 | -1 | 1 | 0 | -1 |
| 18 | 0 | -1 | -1 | 1 | 18 | 0 | -1 | -1 | 0 |
| 19 | 0 | -1 | 1 | 0 | 19 | 0 | -1 | 0 | -1 |
| 20 | 0 | 0 | 0 | -1 | 20 | 0 | 0 | -1 | -1 |
| 21 | 0 | 0 | 0 | 1 | 21 | 0 | 0 | 1 | 1 |
| 22 | 0 | 1 | -1 | 0 | 22 | 0 | 1 | 0 | 1 |
| 23 | 0 | 1 | 1 | -1 | 23 | 0 | 1 | 1 | 0 |
| 24 | 1 | -1 | 0 | 0 | 24 | 1 | -1 | 0 | 1 |
| 25 | 1 | 0 | -1 | -1 | 25 | 1 | 0 | 0 | 0 |
| 26 | 1 | 0 | 1 | 0 | 26 | 1 | 0 | 1 | -1 |
| 27 | 1 | 1 | 0 | 1 | 27 | 1 | 1 | -1 | 0 |

Table 13: The two follow-up designs of size 14 to the factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) that yield augmented designs that are only A- and E-optimum with respect to the second-order model and satisfy the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 14 | -1 | -1 | 0 | 0 | 14 | -1 | -1 | 1 | 0 |
| 15 | -1 | 0 | -1 | 0 | 15 | -1 | 0 | -1 | 0 |
| 16 | -1 | 0 | 1 | 1 | 16 | -1 | 0 | 0 | -1 |
| 17 | -1 | 1 | 0 | -1 | 17 | -1 | 1 | 0 | 1 |
| 18 | 0 | -1 | -1 | 0 | 18 | 0 | -1 | -1 | 1 |
| 19 | 0 | -1 | 1 | -1 | 19 | 0 | -1 | 0 | 0 |
| 20 | 0 | 0 | 0 | -1 | 20 | 0 | 0 | -1 | -1 |
| 21 | 0 | 0 | 0 | 1 | 21 | 0 | 0 | 1 | 1 |
| 22 | 0 | 1 | -1 | 1 | 22 | 0 | 1 | 0 | 0 |
| 23 | 0 | 1 | 1 | 0 | 23 | 0 | 1 | 1 | -1 |
| 24 | 1 | -1 | 0 | 1 | 24 | 1 | -1 | 0 | -1 |
| 25 | 1 | 0 | -1 | -1 | 25 | 1 | 0 | 0 | 1 |
| 26 | 1 | 0 | 1 | 0 | 26 | 1 | 0 | 1 | 0 |
| 27 | 1 | 1 | 0 | 0 | 27 | 1 | 1 | -1 | 0 |

Table 14: The first two follow-up designs of size 14 to the factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) that yield augmented designs that are only G-optimum with respect to the second-order model and satisfy the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 14 | -1 | -1 | 0 | 0, | 14 | -1 | -1 | 0 | 0 |
| 15 | -1 | 0 | 0 | 1 | 15 | -1 | 0 | -1 | 0 |
| 16 | -1 | 0 | 1 | 0 | 16 | -1 | 0 | 0 | -1 |
| 17 | -1 | 1 | -1 | -1 | 17 | -1 | 1 | 1 | 1 |
| 18 | 0 | -1 | -1 | 0 | 18 | 0 | -1 | 0 | 1 |
| 19 | 0 | -1 | 0 | -1 | 19 | 0 | -1 | 1 | 0 |
| 20 | 0 | 0 | -1 | 1 | 20 | 0 | 0 | -1 | 1 |
| 21 | 0 | 0 | 1 | -1 | 21 | 0 | 0 | 1 | -1 |
| 22 | 0 | 1 | 0 | 1 | 22 | 0 | 1 | -1 | 0 |
| 23 | 0 | 1 | 1 | 0 | 23 | 0 | 1 | 0 | -1 |
| 24 | 1 | -1 | 1 | 1 | 24 | 1 | -1 | -1 | -1 |
| 25 | 1 | 0 | -1 | 0 | 25 | 1 | 0 | 0 | 1 |
| 26 | 1 | 0 | 0 | -1 | 26 | 1 | 0 | 1 | 0 |
| 27 | 1 | 1 | 0 | 0 | 27 | 1 | 1 | 0 | 0 |

Table 15: The second two follow-up designs of size 14 to the factors $A_1, A_2, A_3$, and $A_4$ of the DSD(13) that yield augmented designs that are only G-optimum with respect to the second-order model and satisfy the second-order orthogonality criterion.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ |
|---|---|---|---|---|---|---|---|---|---|
| 14 | -1 | -1 | 1 | -1 | 14 | -1 | -1 | -1 | 1 |
| 15 | -1 | 0 | -1 | 0 | 15 | -1 | 0 | 0 | -1 |
| 16 | -1 | 0 | 0 | 1 | 16 | -1 | 0 | 1 | 0 |
| 17 | -1 | 1 | 0 | 0 | 17 | -1 | 1 | 0 | 0 |
| 18 | 0 | -1 | -1 | 0 | 18 | 0 | -1 | 0 | -1 |
| 19 | 0 | -1 | 0 | 1 | 19 | 0 | -1 | 1 | 0 |
| 20 | 0 | 0 | -1 | -1 | 20 | 0 | 0 | -1 | -1 |
| 21 | 0 | 0 | 1 | 1 | 21 | 0 | 0 | 1 | 1 |
| 22 | 0 | 1 | 0 | -1 | 22 | 0 | 1 | -1 | 0 |
| 23 | 0 | 1 | 1 | 0 | 23 | 0 | 1 | 0 | 1 |
| 24 | 1 | -1 | 0 | 0 | 24 | 1 | -1 | 0 | 0 |
| 25 | 1 | 0 | 0 | -1 | 25 | 1 | 0 | -1 | 0 |
| 26 | 1 | 0 | 1 | 0 | 26 | 1 | 0 | 0 | 1 |
| 27 | 1 | 1 | -1 | 1 | 27 | 1 | 1 | 1 | -1 |

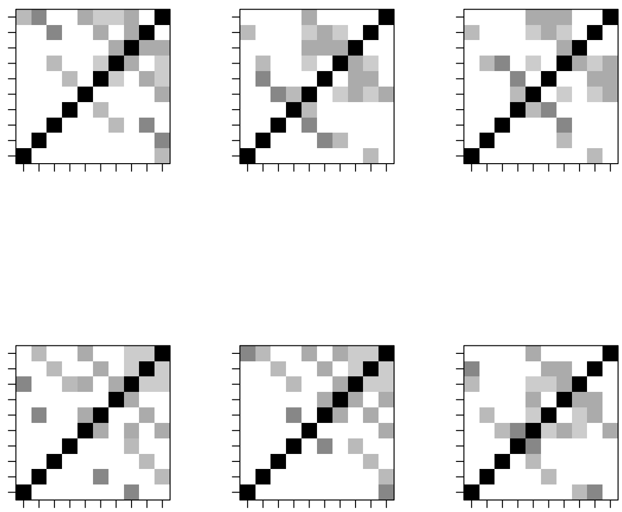

Figure 4: Absolute correlations among second-order contrasts for the A-, D-, and E-optimum augmented designs for the first four factors of the DSD(13).

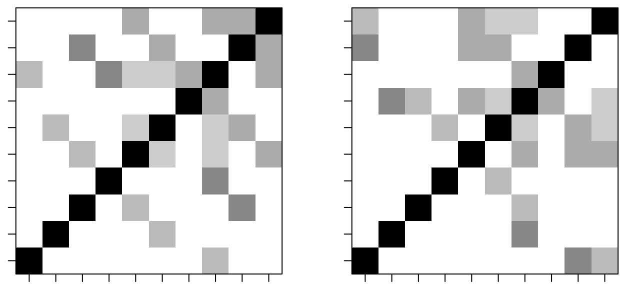

Figure 5: Absolute correlations among second-order contrasts for the A- and E-optimum augmented designs for the first four factors of the DSD(13).

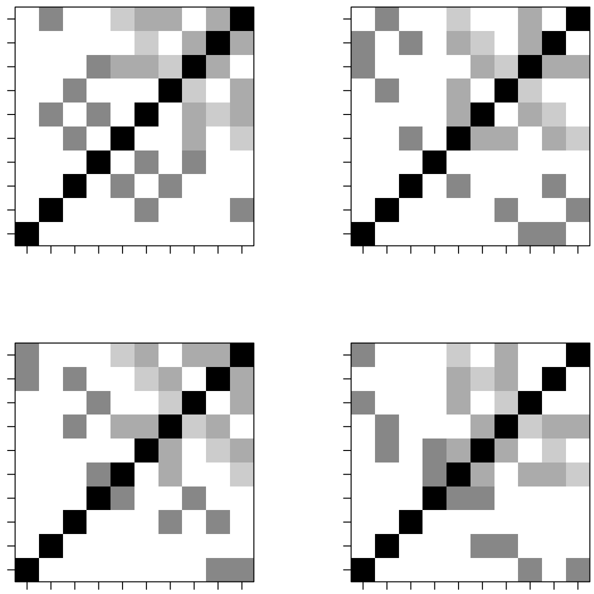

Figure 6: Absolute correlations among second-order contrasts for the G-optimum augmented designs for the first four factors of the DSD(13).

Table 16: Two follow-up designs to the DSD(13) that yield augmented designs satisfying the second-order orthogonality criterion for all factors.

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|
| 14 | 1 | 1 | 1 | 0 | 0 | -1 |
| 15 | 1 | 0 | 0 | 1 | -1 | 1 |
| 16 | 1 | 0 | -1 | 0 | 1 | 0 |
| 17 | 1 | -1 | 0 | -1 | 0 | 0 |
| 18 | 0 | 1 | 0 | 0 | 1 | 1 |
| 19 | 0 | 1 | -1 | -1 | -1 | 0 |
| 20 | 0 | 0 | 1 | -1 | 0 | 1 |
| 21 | 0 | 0 | -1 | 1 | 0 | -1 |
| 22 | 0 | -1 | 1 | 1 | 1 | 0 |
| 23 | 0 | -1 | 0 | 0 | -1 | -1 |
| 24 | -1 | 1 | 0 | 1 | 0 | 0 |
| 25 | -1 | 0 | 1 | 0 | -1 | 0 |
| 26 | -1 | 0 | 0 | -1 | 1 | -1 |
| 27 | -1 | -1 | -1 | 0 | 0 | 1 |

| Run | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ |
|---|---|---|---|---|---|---|
| 14 | -1 | -1 | 0 | -1 | 1 | 0 |
| 15 | -1 | 0 | -1 | 0 | -1 | 1 |
| 16 | -1 | 0 | 0 | 1 | 0 | -1 |
| 17 | -1 | 1 | 1 | 0 | 0 | 0 |
| 18 | 0 | -1 | 0 | 0 | -1 | -1 |
| 19 | 0 | -1 | 1 | 1 | 0 | 1 |
| 20 | 0 | 0 | -1 | 1 | 1 | 0 |
| 21 | 0 | 0 | 1 | -1 | -1 | 0 |
| 22 | 0 | 1 | -1 | -1 | 0 | -1 |
| 23 | 0 | 1 | 0 | 0 | 1 | 1 |
| 24 | 1 | -1 | -1 | 0 | 0 | 0 |
| 25 | 1 | 0 | 0 | -1 | 0 | 1 |
| 26 | 1 | 0 | 1 | 0 | 1 | -1 |
| 27 | 1 | 1 | 0 | 1 | -1 | 0 |

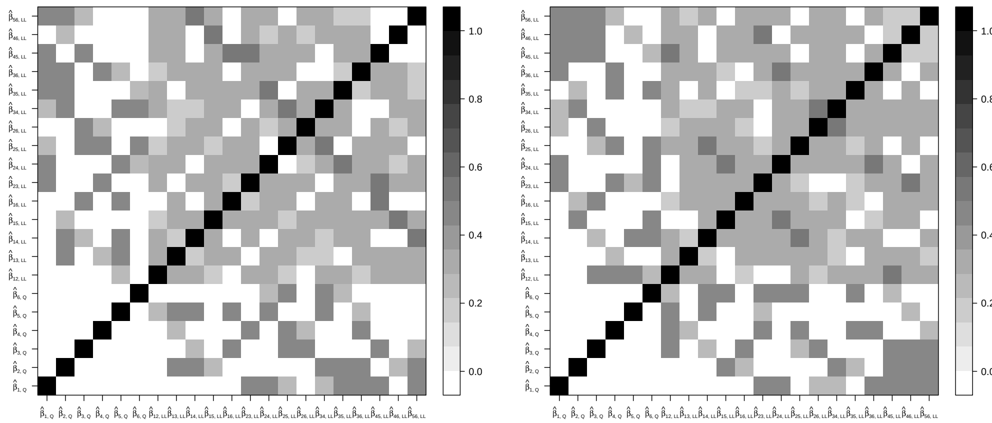

Figure 7: Absolute correlations among second-order contrasts for the designs formed by augmenting the DSD(13) with the follow-up runs in Table 16, respectively.

## References


Jones, B., Nachtsheim, C. J., 2011. A class of three-level designs for definitive screening in the presence of second-order effects. Journal of Quality Technology 43 (1), 1–15.