STAT 525

Chapter 18
ANOVA Diagnostics and Remedies

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Overview

- General assumptions
  - Normally distributed error terms
  - Independent observations
  - Constant variance
- Will adapt diagnostics and remedial measures from linear regression
- Many are the same but others require slight modifications
- Key difference to linear regression: for ANOVA, data are grouped in their covariate values.

Residuals

- Predicted values are the cell means

$$\widehat{\mu}_i = \overline{Y}_{i.}$$

- Residuals are the difference between the observed and predicted

$$e_{ij} = Y_{ij} - \overline{Y}_{i.}$$

- Properties:
  - Same least squares properties
  - $\sum_j e_{ij} = 0$

Basic Plots

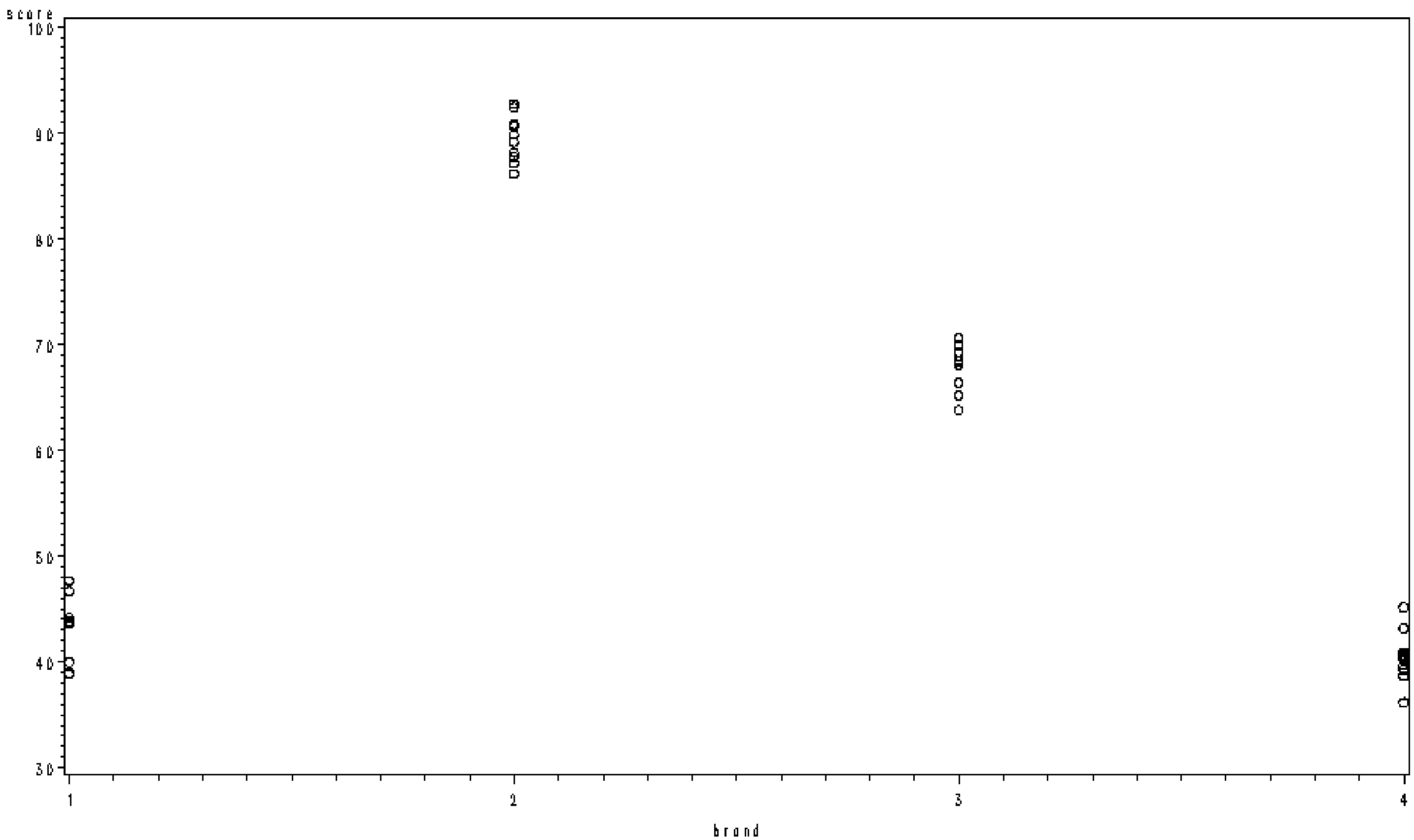
- Plot the data vs the factor levels
- Plot the residuals vs the factor levels
- Plot the residuals vs the fitted values
- Histogram of the residuals
- QQplot of the residuals

Example (Page 777)

- Experiment designed to study the effectiveness of four rust inhibitors
- Forty units were used in the experiment
- Units randomly and equally assigned to rust inhibitors ($n_i = 10$)
- Each unit exposed to severe weather conditions
- Y coded score (higher means less rust)
- X brand of rust inhibitor
  - $i = 1, 2, 3, 4$
  - $j = 1, 2, .., 10$

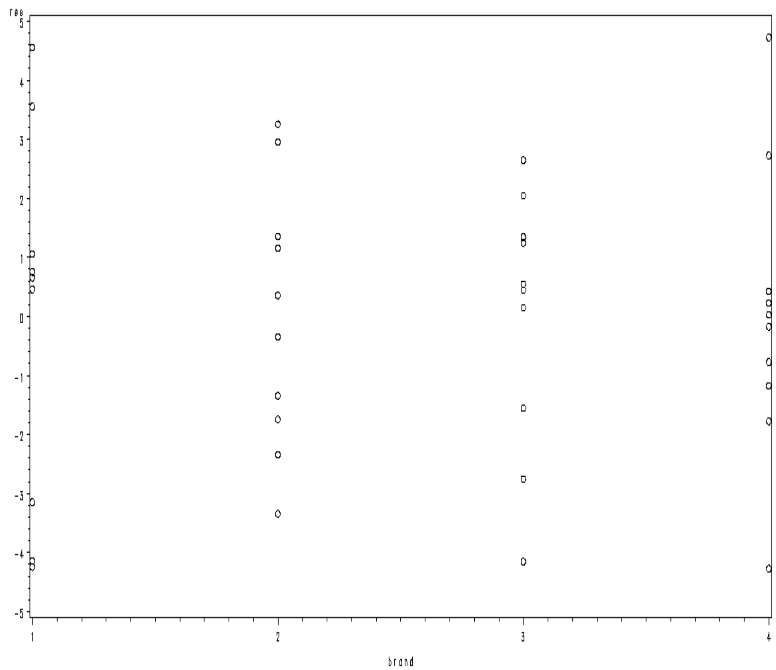
Scatterplot

```
options nocenter; goptions colors=('none');
data a1;
    infile 'u:\.www\datasets525\CH17TA02.txt';
    input score brand;

symbol1 v=circle i=none;
proc gplot data=a1;
    plot score*brand;
run; quit;
```

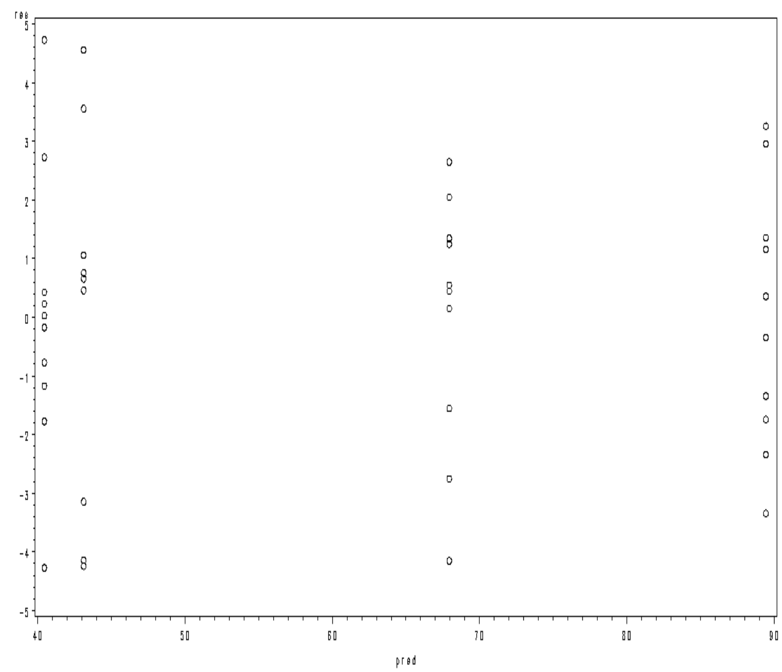


Residual Plots

```
proc glm data=a1;
    class brand;
    model score=brand;
    output out=a2 r=res p=pred;

proc gplot;
    plot res*(brand pred);
run; quit;
```



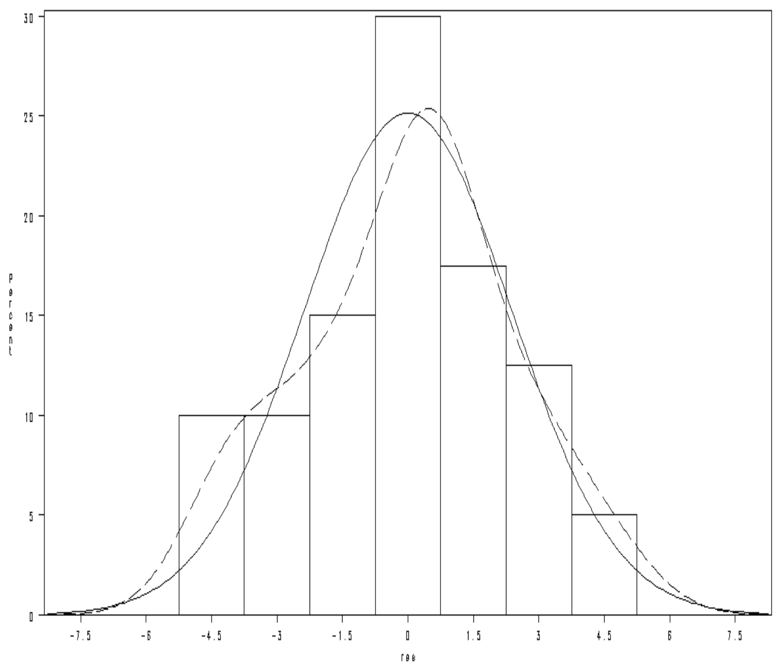
Residual vs. Brand



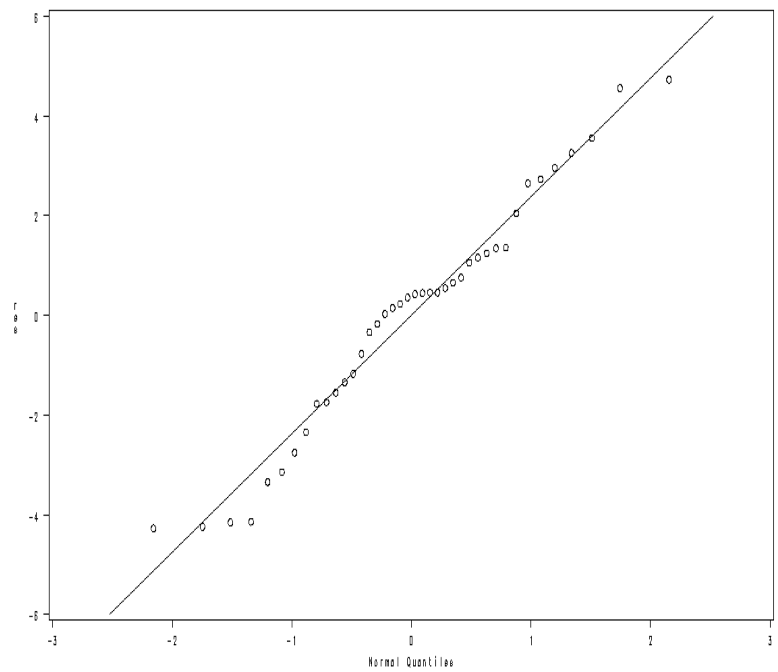
Residual vs. $\widehat{Y}_i$

Histogram & QQPlot

```
proc univariate noprint data=a2;
    histogram res / normal kernel(L=2);
    qqplot res / normal (L=1 mu=est sigma=est);
run; quit;
```



Histogram of Residuals



QQPlot of Residuals

Summary

- Look for
  - Outliers
  - Non-constant variance
  - Non-normal errors
- Can plot residuals vs time or other variables if available
  - Independent observations

Formal Tests

- Normality
  - Wilk-Shapiro
  - Anderson-Darling
  - Kolmogorov-Smirnov
- Homogeneity of Variance
  - Hartley test
  - Modified Levene test (aka Brown-Forsythe test in SAS)
  - Bartlett's

Homogeneity of Variance: Hartley Test

- It requires equal sample sizes across factor levels, i.e., $n_i = n$
- Hartley statistic,

$$H^* = \frac{\max(s_i^2)}{\min(s_i^2)} \sim H(r, n-1), \text{under } H_0$$

- Percentiles of H(r,df) are shown in Table B.10 (p. 1336).

Homogeneity of Variance: Modified Levene Test

- Called Brown-Forsythe test in SAS
- Test statistic,
  - Define $d_{ij} = |Y_{ij} - \tilde{Y}_i|$, with $\tilde{Y}_i$ the median at factor level i
  - Calculate $\bar{d}_{i\cdot} = \sum_j d_{ij}/n_i$, $\bar{d}_{\cdot\cdot} = \sum_i \sum_j d_{ij}/n_T$
  - Calculate

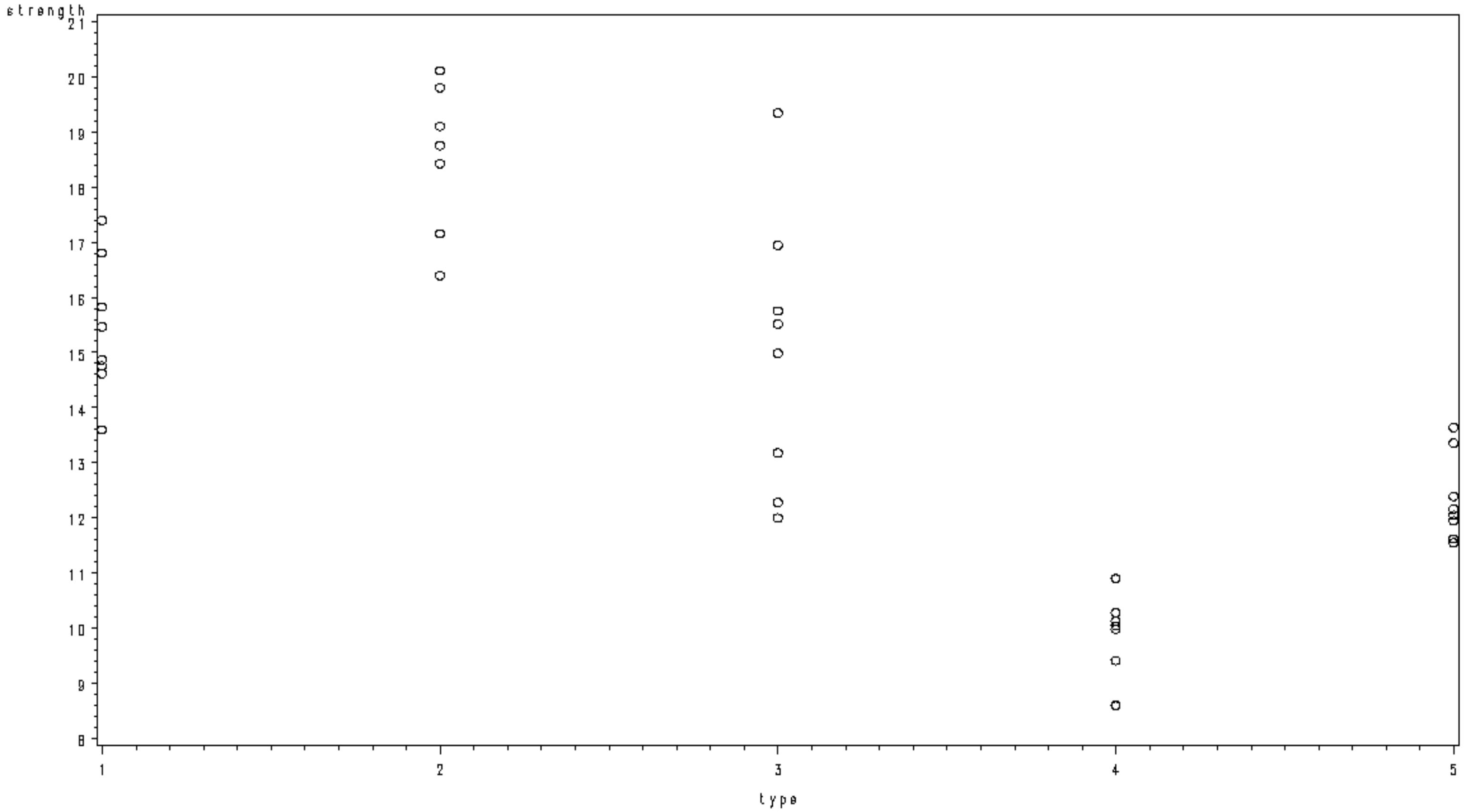
$$\begin{aligned} MSTR &= \sum_i n_i(\bar{d}_{i\cdot} - \bar{d}_{\cdot\cdot})^2/(r-1), \\ MSE &= \sum_i \sum_j (d_{ij} - \bar{d}_{i\cdot})^2/(n_T - r) \end{aligned}$$

  - $F^*_{BF} = MSTR/MSE \overset{approx}{\sim} F(r-1, n_T - r)$ under H_0
- Modified Levene test is often the best choice
  - Unlike the Hartley test, it is robust against departures from normality
  - It does not require equal sample sizes
- In `PROC GLM`, Use option `HOVTEST=BF` for `MEANS` statement

Example (Page 783)

- Experiment designed to assess the strength of five types of flux used in soldering wire boards
- Forty units were used in the experiment
- Units randomly and equally assigned to five types of flux ($n_i = 8$)
- Y – strength
- X – type of flux

```
data a1;
    infile 'u:\.www\datasets525\CH18TA02.DAT';
    input strength type;

/* Scatterplot */
proc gplot data=a1;
    plot strength*type;
run; quit;
```



```
/* Modified Levene Test */
proc glm data=a1;
    class type;
    model strength=type;
    means type / hovtest=bf clm;
run; quit;
```

Brown and Forsythe's Test for Homogeneity of strength Variance
ANOVA of Absolute Deviations from Group Medians

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
type	4	9.3477	2.3369	2.94	0.0341
Error	35	27.8606	0.7960		

Level of type	N	----------strength---------- Mean	Std Dev
1	8	15.4200000	1.23713956
2	8	18.5275000	1.25297076
3	8	15.0037500	2.48664397
4	8	9.7412500	0.81660337
5	8	12.3400000	0.76941536

Remedies

- Delete potential outliers
  - Is their removal important?
- Use weighted regression
- Box-Cox Transformation
- Non-parametric procedures

Variance Stabilization Transformations

- Consider response Y with E(Y)=μ_x and Var(Y)=$\sigma_x^2 = g(\mu_x)$
  - σ_x^2 depends on μ_x
- Want to find $\tilde{Y} = f(Y)$ such that Var($\tilde{Y}$)$\approx c$
  - What are the mean and var of $\tilde{Y}$?

Delta Method

Consider $f(Y)$ where $f'(\mu_x) \neq 0$

$$f(Y) \approx f(\mu_x) + (Y - \mu_x) f'(\mu_x)$$

$$\text{E}(\tilde{Y}) = \text{E}(f(Y)) \approx \text{E}(f(\mu_x)) + \text{E}((Y - \mu_x) f'(\mu_x)) = f(\mu_x)$$

$$\text{Var}(\tilde{Y}) \approx [f'(\mu_x)]^2 \text{Var}(Y) = [f'(\mu_x)]^2 \sigma_x^2$$

- Want to choose f such that $[f'(\mu_x)]^2 g(\mu_x) \approx c$

Examples

$g(\mu) = \mu$	(Poisson)	$f(\mu) = \int \frac{1}{\sqrt{\mu}} d\mu \to f(Y) = \sqrt{Y}$
$g(\mu) = \mu(1-\mu)$	(Binomial)	$f(\mu) = \int \frac{1}{\sqrt{\mu(1-\mu)}} d\mu \to f(X) = \arcsin(\sqrt{Y})$
$g(\mu) = \mu^{2\beta}$	(Box-Cox)	$f(\mu) = \int \mu^{-\beta} d\mu \to f(Y) = Y^{1-\beta}$
$g(\mu) = \mu^2$	(Box-Cox)	$f(\mu) = \int \frac{1}{\mu} d\mu \to f(Y) = \log X$

Transformation Guides

- Regress $\log(s_i)$ vs $\log(\overline{Y_{i.}})$ $\rightarrow$ $\widehat{\lambda}$ =1-slope for $\tilde{Y} = Y^\lambda$
  - $f(\mu) = \mu^\lambda \Longrightarrow \log \sqrt{g(\mu)} = -\log \lambda + (1-\lambda) \log \mu$
  - When $\sigma_i^2 \propto \mu_i$ use $\sqrt{\ }$
  - When $\sigma_i \propto \mu_i$ use log
  - When $\sigma_i \propto \mu_i^2$ use $1/Y$
- For proportions, use arcsin($\sqrt{\ }$)
  - Use `arsin(sqrt(Y))` in SAS data step

Example (Page 783)

```
proc transreg data=a1;
    model boxcox(strength)=class(type);
run; quit;
```

```
  Lambda      R-Square    Log Like
   -1.50          0.86    -15.3143
   -1.25          0.86    -14.2378 *
   -1.00          0.86    -13.4223 *
   -0.75          0.86    -12.8608 *
   -0.50          0.85    -12.5428 *
   -0.25          0.85    -12.4549 <
    0.00 +        0.85    -12.5819 *
    0.25          0.84    -12.9078 *
    0.50          0.84    -13.4163 *
    0.75          0.83    -14.0919 *
    1.00          0.83    -14.9199
    1.25          0.82    -15.8868
    1.50          0.81    -16.9807

< - Best Lambda
* - Confidence Interval
+ - Convenient Lambda
```

- Log-transformation is suggested here.
- May also explore the relationship between s_i vs $\bar{Y}_{i.}$ as shown on P.790-791.

Nonparametric Approach

- Based on ranking the observations and using the ranks
  - Rank Y_{ij} in ascending order from 1 to n_T, i.e., R_{ij}
  - Specify the score $d_{ij} = d(R_{ij})$
  - Apply One-Way ANOVA to d_{ij}, $1 \leq j \leq n_i$, $1 \leq i \leq r$
- Wilcoxon Scores, $d(R_{ij}) = R_{ij}$
  - Produces the Kruskal-Wallis test in the one-way ANOVA
  - Produces the Man-Whitney-Wilcoxon test for two-sample data ($r = 2$)
- Median scores, $d(R_{ij}) = 1[R_{ij} > (n_T + 1)/2]$
  - Produces the Brown-Mood test in the one-way ANOVA
  - Produces the median test for two-sample data ($r = 2$)
- SAS procedure `PROC NPAR1WAY`

Example (Page 783)

```
proc npar1way data=a1 median wilcoxon;
    class type;
    var strength;
run; quit;
```

```
         Wilcoxon Scores (Rank Sums) for Variable strength
                    Classified by Variable type

                        Sum of      Expected       Std Dev          Mean
type          N         Scores      Under H0      Under H0         Score
--------------------------------------------------------------------
1             8          201.0         164.0     29.573377       25.1250
2             8          282.0         164.0     29.573377       35.2500
3             8          190.0         164.0     29.573377       23.7500
4             8           36.0         164.0     29.573377        4.5000
5             8          111.0         164.0     29.573377       13.8750

                 Average scores were used for ties.

   Kruskal-Wallis Test

Chi-Square          32.1634
DF                        4
Pr > Chi-Square      <.0001
```

Median Scores (Number of Points Above Median) for Variable strength
Classified by Variable type

type	N	Sum of Scores	Expected Under H0	Std Dev Under H0	Mean Score
1	8	7.0	4.0	1.281025	0.8750
2	8	8.0	4.0	1.281025	1.0000
3	8	5.0	4.0	1.281025	0.6250
4	8	0.0	4.0	1.281025	0.0000
5	8	0.0	4.0	1.281025	0.0000

Average scores were used for ties.

Median One-Way Analysis

Chi-Square	28.2750
DF	4
Pr > Chi-Square	<.0001

- χ^2-distributions, instead of F-distributions, are used due to the fact that the error variances are known in theory.
- Exact tests can be taken using the statement `EXACT MEDIAN WILCOXON;`
  - Recommended for small, sparse, skewed, or heavily tied dataset.

Chapter Review

- Diagnostics
  - Plots
  - Residual checks
  - Formal Tests
- Remedial Measures