Assignment 8 Answer Keys

Problem 1

(a) This is a factorial design with two factors (glass type and temperature) and each factor having three levels. The statistical model for it is

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \epsilon_{ijk}, \qquad i = 1,2,3, j = 1,2,3, k = 1,2,3,$$

where μ is the grand mean, τ_i are main effects for glass type i, β_j main effects for temperature j, $(\tau\beta)_{ij}$ interaction effects of temperature i and temperature j, ϵ_{ijk} are iid $N(0, \sigma^2)$ random variables. Also, we have constraints

$$\sum_i \tau_i = \sum_j \beta_j = \sum_i (\tau\beta)_{ij} = \sum_j (\tau\beta)_{ij} = 0.$$

The ANOVA table output from SAS is shown below (I replaced the line for the model SS by lines for the factorial effects).

Source	DF	Sum of Squares	Mean Square	F Value	Pr > F
gls	2	310.88963	155.44481	35.81	<.0001
tmp	2	18142.45630	9071.22815	2089.97	<.0001
gls*tmp	4	642.40593	160.60148	37.00	<.0001
Error	18	78.12667	4.34037		
Corrected Total	26	19173.87852			

Since the p-values for all the three factorial effects are less than 0.0001, I conclude that all the involved factorial effects are significant.

(b) The following table summarizes the cell means $\bar{y}_{ij\cdot}$, the group means $\bar{y}_{i\cdot\cdot}, \bar{y}_{\cdot j\cdot}$ and the the overall mean $\bar{y}_{\cdots}$.

	$j=1$	$j=2$	$j=3$	$\bar{y}_{i\cdot\cdot}$
$i=1$	57.267	106.733	128.600	91.533
$i=2$	55.300	105.167	114.633	91.700
$i=3$	57.333	107.467	103.667	89.489
$\bar{y}_{\cdot j\cdot}$	56.633	106.456	115.633	$\bar{y}_{\cdots} = 92.907$

Then using the following estimating formulae

$$\begin{aligned}
\hat{\mu} &= \bar{y}_{\cdots}, \\
\hat{\tau}_i &= \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdots}, i = 1,2,3, \\
\hat{\beta}_j &= \bar{y}_{\cdot j\cdot} - \bar{y}_{\cdots}, j = 1,2,3, \\
(\widehat{\tau\beta})_{ij} &= \bar{y}_{ij\cdot} - \bar{y}_{i\cdot\cdot} - \bar{y}_{\cdot j\cdot} + \bar{y}_{\cdots}, i = 1,2,3, j = 1,2,3,
\end{aligned}$$

I get $\hat{\mu} = 92.907$, and other estimates summarized in the following table.

$\hat{\tau}_1$	$\hat{\tau}_2$	$\hat{\tau}_3$
4.626	-1.207	-3.419

$\hat{\beta}_1$	$\hat{\beta}_2$	$\hat{\beta}_3$
-36.274	13.548	22.726

$\widehat{(\tau\beta)}_{ij}$	$j=1$	$j=2$	$j=3$
$i=1$	-3.993	-4.348	8.341
$i=2$	-0.126	-0.081	0.207
$i=3$	4.119	4.430	-8.548

(c) The diagnostic plots in Figure 1 are: normal probability Q-Q plot, plot of residuals versus the response, plot of residuals versus glass type (row block), plot of residuals versus temperature (column block), and plot of residuals versus predicted values.

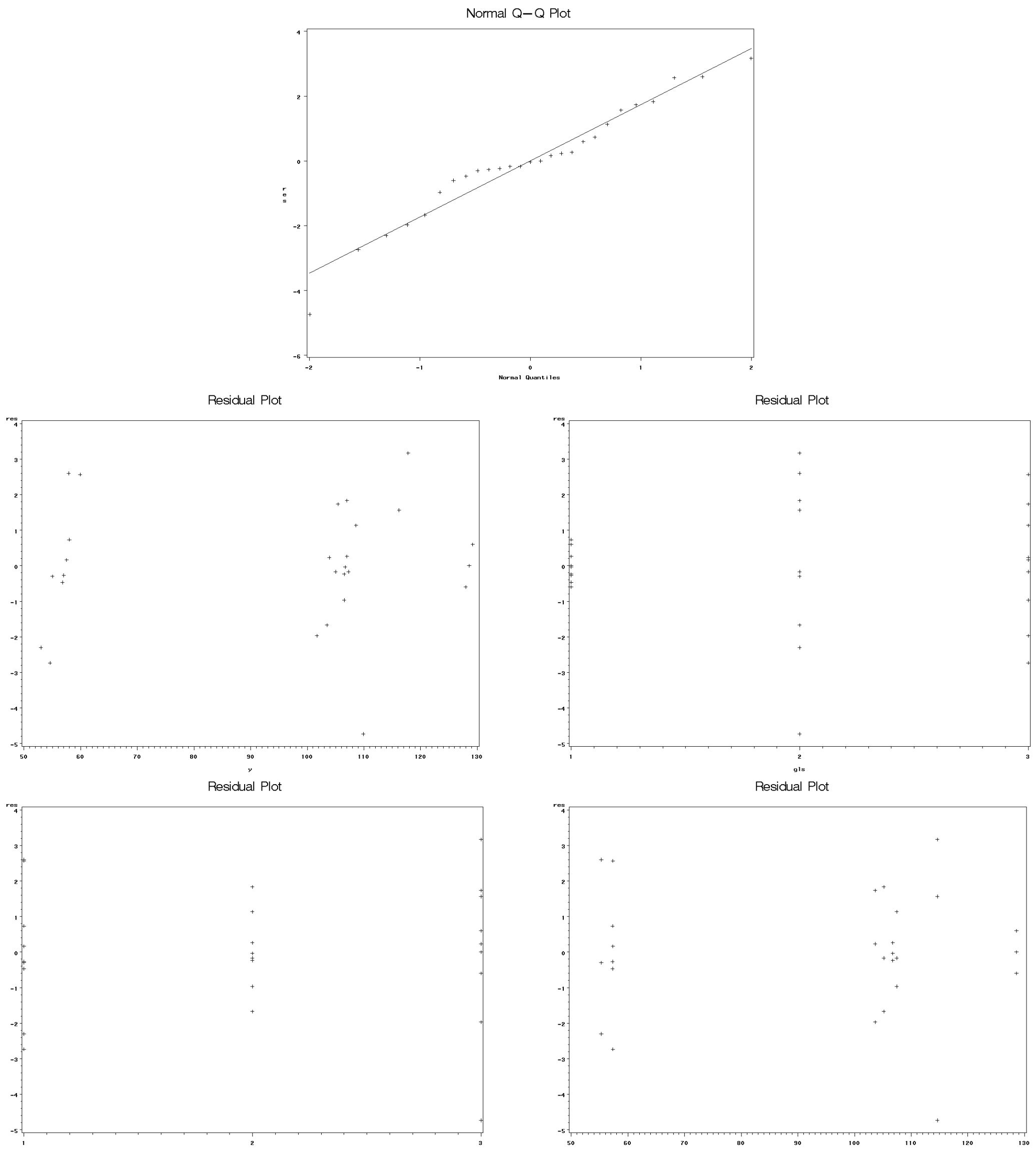

Figure 1: Diagnostic Plots

The normal Q-Q plot shows that the normality assumption is valid. None of the four residual plots has shown unequal variances or potential outliers.

(d) The two interaction plots are generated as below. The first one uses temperature as the horizontal axis, while the second one uses glass type.

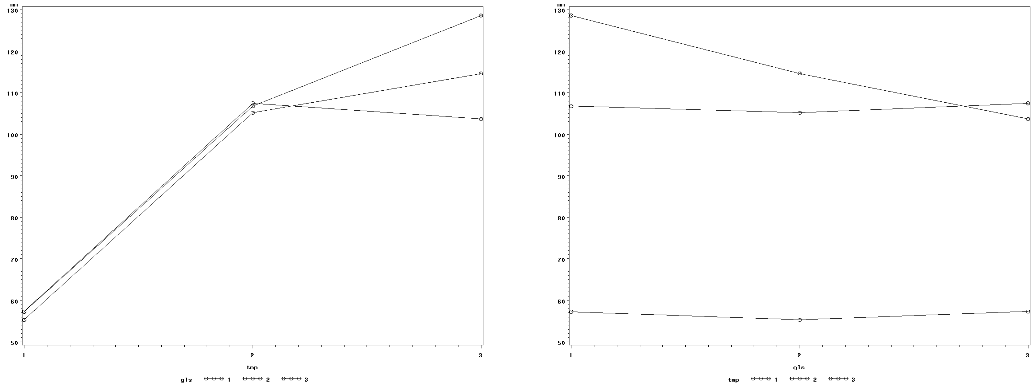

Figure 2: Interaction Plots

(e) The Bonferroni procedure result for pairwise comparision of glass type level means is as shown below.

Bonferroni (Dunn) t Tests for y

NOTE: This test controls the Type I experimentwise error rate, but it generally has a higher Type II error rate than REGWQ.

Alpha	0.05
Error Degrees of Freedom	18
Error Mean Square	4.34037
Critical Value of t	2.63914
Minimum Significant Difference	2.5919

Means with the same letter are not significantly different.

Bon Grouping	Mean	N	gls
A	97.5333	9	1
B	91.7000	9	2
B			
B	89.4889	9	3

Hence, the difference between level means of glass type 2 and 3 is not significant. But the level mean of glass type 1 is significantly different from the other two.

(f) The Tukey method result for pairwise comparison between cell means is as shown below.

Adjustment for Multiple Comparisons: Tukey

gls	tmp	y LSMEAN	LSMEAN Number
1	1	57.266667	1
1	2	106.733333	2
1	3	128.600000	3
2	1	55.300000	4
2	2	105.166667	5
2	3	114.633333	6
3	1	57.333333	7
3	2	107.466667	8
3	3	103.666667	9

```
                     Least Squares Means for Effect gls*tmp
                    t for H0: LSMean(i)=LSMean(j) / Pr > |t|

                             Dependent Variable: y

i/j          1          2          3          4          5          6          7          8          9

   1             -29.08   -41.9348   1.156147    -28.159   -33.7242   -0.03919   -29.5111   -27.2772
                 <.0001     <.0001     0.9561     <.0001     <.0001     1.0000     <.0001     <.0001
   2  29.08003             -12.8548   30.23618   0.920998   -4.64418   29.04084   -0.43111   1.802805
        <.0001               <.0001     <.0001     0.9886     0.0049     <.0001     0.9999     0.6802
   3  41.93482   12.85479              43.09096   13.77578   8.210602   41.89563   12.42368   14.65759
        <.0001     <.0001                <.0001     <.0001     <.0001     <.0001     <.0001     <.0001
   4  -1.15615   -30.2362    -43.091              -29.3152   -34.8804   -1.19534   -30.6673   -28.4334
        0.9561     <.0001     <.0001                <.0001     <.0001     0.9474     <.0001     <.0001
   5  28.15903     -0.921   -13.7758   29.31518              -5.56518   28.11984    -1.3521   0.881807
        <.0001     0.9886     <.0001     <.0001                0.0007     <.0001     0.9015     0.9913
   6  33.72422   4.644183    -8.2106   34.88036   5.565181              33.68502   4.213077   6.446988
        <.0001     0.0049     <.0001     <.0001     0.0007                <.0001     0.0120     0.0001
   7  0.039191   -29.0408   -41.8956   1.195338   -28.1198    -33.685              -29.4719    -27.238
        1.0000     <.0001     <.0001     0.9474     <.0001     <.0001                <.0001     <.0001
   8  29.51114   0.431106   -12.4237   30.66728   1.352104   -4.21308   29.47195              2.233911
        <.0001     0.9999     <.0001     <.0001     0.9015     0.0120     <.0001                0.4261
   9  27.27723   -1.80281   -14.6576   28.43337   -0.88181   -6.44699   27.23804   -2.23391
        <.0001     0.6802     <.0001     <.0001     0.9913     0.0001     <.0001     0.4261
```

We have two groups of cells that any pairs within each group have insignificantly different means. These two groups are $\{(1,1),(2,1),(3,1)\}$ and $\{(1,2),(2,2),(3,2),(3,3)\}$. Any other pair of cells have significantly different means.

(g) The glass type effects at different temperature levels are shown below.

```
                        gls*tmp Effect Sliced by tmp for y

                                 Sum of
        tmp          DF         Squares      Mean Square     F Value     Pr > F

        1             2        8.006667         4.003333        0.92     0.4156
        2             2        8.282222         4.141111        0.95     0.4038
        3             2      937.006667       468.503333      107.94     <.0001
```

The p-value at temperature level 3 (temperature = 150) is less than 0.0001 and p-values at the other two levels are greater than 0.4. This verifies the observation that the glass types have different effects on the response only when the temperature is at 150.

(h) First, I convert the categorical variable glass type to dummy variables x_1 and x_2 by

Glass Type	x_1	x_2
1	1	0
2	0	1
3	-1	-1

Second, I standardize the temperature variable by

$$t = (\text{temperature} - 125)/25.$$

Then the model I fit is

$$y_{ijk} = \beta_0 + \beta_1 x_1 + \beta_2 x_2 + \beta_3 t + \beta_4 x_1 t + \beta_5 x_2 t + \beta_6 t^2 + \beta_7 x_1 t^2 + \beta_8 x_2 t^2 + \epsilon_{ijk},$$

where ϵ_{ijk} are iid $N(0, \sigma^2)$. The parameter estimates are shown below.

Variable	DF	Parameter Estimate	Standard Error	t Value	Pr > |t|
Intercept	1	106.45556	0.69445	153.29	<.0001
x1	1	0.27778	0.98210	0.28	0.7805
x2	1	-1.28889	0.98210	-1.31	0.2059
t	1	29.50000	0.49105	60.08	<.0001
x1t	1	6.16667	0.69445	8.88	<.0001
x2t	1	0.16667	0.69445	0.24	0.8130
t2	1	-20.32222	0.85053	-23.89	<.0001
x1t2	1	6.52222	1.20283	5.42	<.0001
x2t2	1	0.12222	1.20283	0.10	0.9202

Hence I get the following three response curves, where $t = (\text{temperature} - 125)/25$:

- Glass type 1 ($x_1 = 1, x_2 = 0$):

$$\begin{aligned} E(y_{1t}) &= (106.46 + 0.28) + (29.50 + 6.17)t + (-20.32 + 6.52)t^2 \\ &= 106.74 + 35.67t - 13.8t^2. \end{aligned}$$

- Glass type 2 ($x_1 = 0, x_2 = 1$):

$$\begin{aligned} E(y_{2t}) &= (106.46 - 1.29) + (29.50 + 0.17)t + (-20.32 + 0.12)t^2 \\ &= 105.17 + 29.67t - 20.2t^2. \end{aligned}$$

- Glass type 3 ($x_1 = -1, x_2 = -1$):

$$\begin{aligned} E(y_{3t}) &= (106.46 - 0.28 + 1.29) + (29.50 - 6.17 - 0.17)t + (-20.32 - 6.52 - 0.12)t^2 \\ &= 107.47 + 23.16t - 26.96t^2. \quad \blacksquare \end{aligned}$$