STAT 517 HW2 solution:

6.3.1 Note that under θ, the random variable $(\theta_0/\theta)(2/\theta_0)\sum_{i=1}^n X_i$ has a $\chi^2(2n)$ distribution. Therefore, the power function is

$$\gamma(\theta) = P\left[T \le \frac{\theta_0}{\theta}\chi^2_{1-\alpha/2}(2n)\right] + P\left[T \ge \frac{\theta_0}{\theta}\chi^2_{\alpha/2}(2n)\right],$$

where T has a $\chi^2(2n)$ distribution.

6.3.3 The decision rule (6.3.6) is equivalent to the decision rule

$$\text{Reject } H_0 \text{ if } |z| \ge z_{\alpha/2},$$

where $z = (\overline{x} - \theta_0)/(\sigma/\sqrt{n})$. The power function is

$$\begin{aligned}\gamma(\theta) &= P_\theta\left[\left|\frac{\overline{X}-\theta_0}{\sigma/\sqrt{n}}\right| \ge z_{\alpha/2}\right] \\ &= P_\theta\left[\left|\frac{\overline{X}-\theta}{\sigma/\sqrt{n}}\right| \le -z_{\alpha/2} + \frac{\sqrt{n}(\theta_0-\theta)}{\sigma}\right] \\ &\quad + P_\theta\left[\left|\frac{\overline{X}-\theta}{\sigma/\sqrt{n}}\right| \ge z_{\alpha/2} + \frac{\sqrt{n}(\theta_0-\theta)}{\sigma}\right] \\ &= \Phi\left[-z_{\alpha/2} + \frac{\sqrt{n}(\theta_0-\theta)}{\sigma}\right] + 1 - \Phi\left[z_{\alpha/2} + \frac{\sqrt{n}(\theta_0-\theta)}{\sigma}\right].\end{aligned}$$

6.3.8 Part (a). Under Ω, the mle is $\overline{x}$. After simplification, the likelihood ratio test is

$$\Lambda = e^{-\theta_0} e^{\overline{x} - n\overline{x}\log(\overline{x}/\theta_0)}.$$

Treating Λ as a function of $\overline{x}$, upon differentiating it twice we see that the function has a positive real critical value which is a maximum. Hence, the likelihood ratio test is equivalent to rejecting H_0, if $Y \le c_1$ or $Y \ge c_2$ where $Y = n\overline{X}$. Under H_0, Y has a Poisson distribution with mean $n\theta_0$. The significance level of the test is 0.056 for the situation described in Part (b).

6.3.15 The likelihood function can be expressed as

$$L(\theta) = \theta^{n\overline{x}}(1-\theta)^{n-n\overline{x}}.$$

To get the information, note that

$$\log p(x;\theta) = x\log\theta + (1-x)\log(1-\theta).$$

Upon taking the first two partial derivatives with respect to θ, we obtain the information

$$I(\theta) = E\left[\frac{X}{\theta^2}\right] - E\left[\frac{1-X}{1-\theta^2}\right] = \frac{1}{\theta(1-\theta)}.$$

(a). Under Ω, the mle is $\overline{x}$. Hence, the likelihood ratio test statistic is

$$\Lambda = \left(\frac{1}{3\overline{x}}\right)^{n\overline{x}} \left(\frac{2/3}{1-\overline{x}}\right)^{n-n\overline{x}}.$$

(b). Wald's test statistic is

$$\chi_W^2 = \left[\frac{\overline{x}-(1/3)}{\sqrt{\overline{x}(1-\overline{x})/n}}\right]^2.$$

(c). For the scores test,

$$l'(\theta_0) = \sum_{i=1}^{n}\left[\frac{x_i}{\theta} - \frac{1-x_i}{1-\theta}\right] = \frac{n(\overline{x}-\theta)}{\theta(1-\theta)}.$$

Hence, the scores test statistic is

$$\chi_R^2 = \left\{\frac{n(\overline{x}-\theta_0)}{\theta_0(1-\theta_0)}\Big/\sqrt{\frac{n}{\theta_0(1-\theta_0)}}\right\}^2 = \left\{\frac{\sqrt{n}(\overline{x}-\theta_0)}{\sqrt{\theta_0(1-\theta_0)}}\right\}^2.$$

6.4.5 $L = \left(\frac{1}{2\rho}\right)^n$, provided $\theta - \rho \le y_1 \le y_n \le \theta + \rho$. To maximize L make ρ as small as possible which is accomplished by setting

$$\hat{\theta} - \hat{\rho} = Y_1 \text{ and } \hat{\theta} + \hat{\rho} = Y_n.$$

So

$$\hat{\theta} = \frac{Y_1 + Y_n}{2} \text{ and } \hat{\rho} = \frac{Y_n - Y_1}{2}.$$

Thus

$$E\left[\frac{(n+1)Y_n}{n}\right] = \theta, \quad \text{Var}\left[\frac{(n+1)Y_n}{n}\right] = \frac{\theta^2}{n(n+2)}.$$

However, we have that

$$\frac{\theta^2}{n(n+2)} < \frac{\theta^2}{n} = \frac{1}{nE\left\{\left[\frac{\partial \log f(X;\theta)}{\partial\theta}\right]^2\right\}},$$

which seems like a contradiction to the Rao-Cramér inequality until we recognize that this is not a regular case.

6.4.8 Because $b > 0$,

$$P(X_i \le t) = P\left(e_i \le \frac{t-a}{b}\right),$$

from which the result follows.

6.4.10 Write I_{12} as

$$I_{12} = \frac{1}{b^2} \int_{-\infty}^{\infty} \{z\} \left\{ \frac{f'(z)}{f(z)} \right\}^2 \{f(z)\}\, dz.$$

Note that the function in the first set of braces is odd while the last two functions are even (the third because of the assumed symmetry). Thus their product is an odd function and hence the integral of it from $-\infty$ to ∞ is 0.