Solution HW1:

6.1.2

(a)
Set the derivative w.r.t. θ as 0. Then we can get the MLE of $\theta\ \ as\ \frac{-n}{\sum_{i=1}^{n}\log(x\_i)}$.

(b)
Set the derivative w.r.t. θ as 0. Then we can get the MLE of θ as min{x_i}

6.1.5
Set the derivative w.r.t. θ as 0. Then MLE of $\hat{\theta}$ is equal to $\frac{\sum_{i=1}^{n} x\_i}{n}$.
P_hat = 1 - exp (-2 /$\hat{\theta}$)

6.1.10

(a)
Set the derivative w.r.t. θ as 0. Then the MLE is $\hat{\theta} = \frac{\sum_{i=1}^{n} x\_i}{n}$

(b)
Set the derivative w.r.t. θ as 0. Then MLE is $\hat{\theta}$ =max{0, $\frac{\sum_{i=1}^{n} x\_i}{n}$}

6.2.2

6.2.2 $\frac{\partial \log f(x;\theta)}{\partial\theta} = \frac{-1}{\theta};\quad nE\left[\left(\frac{-1}{\theta}\right)^2\right] = \frac{n}{\theta^2}$.
Also

$$E(Y_n) = \int_0^\theta (ny^n/\theta^n)\,dy = \frac{n}{n+1}\theta,$$
$$\mathrm{Var}(Y_n) = \frac{n}{n+2}\theta^2 - \left(\frac{n}{n+1}\right)^2\theta^2 = \frac{n\theta^2}{(n+2)(n+1)^2}.$$

6.2.8

6.2.8 (a).

$$
\begin{aligned}
\log f(x;\theta) &= -\frac{1}{2}\log(2\pi) - \frac{1}{2}\log\theta - \frac{x^2}{2\theta}, \\
\frac{\partial \log f(x;\theta)}{\partial\theta} &= -\frac{1}{2\theta} + \frac{x^2}{2\theta^2}, \\
\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2} &= \frac{1}{2\theta^2} - \frac{x^2}{\theta^3}, \\
-nE\left[\frac{\partial^2 \log f(x;\theta)}{\partial\theta^2}\right] &= \frac{-n}{2\theta^2} + \frac{n}{\theta^2} = \frac{n}{2\theta^2} = nI(\theta).
\end{aligned}
$$

(b). Here $\hat{\theta} = \sum X_i^2/n$. Since $\sum X_i^2/\theta$ is $\chi^2(n)$, we have

$$\mathrm{Var}(\hat{\theta}) = \frac{\theta^2}{n^2}\,\mathrm{Var}\left(\frac{\sum X_i^2}{\theta}\right) = \frac{2\theta^2}{n} = \frac{1}{nI(\theta)}.$$

6.2.10

6.2.10 Note that

$$
\begin{aligned}
E[|X_1|] &= 2\int_0^\infty \frac{x}{\sqrt{2\pi}}\exp\left\{-\frac{1}{2}\frac{x^2}{\theta}\right\}dx \\
&= 2\sqrt{\theta}\int_0^\infty \frac{1}{\sqrt{2\pi}}\exp\{-z\}\,dz = \sqrt{\frac{2}{\pi}}\sqrt{\theta}.
\end{aligned}
$$

So $c = \sqrt{\pi/2}/n$. Hence, $Y = n^{-1}\sum_{i=1}^n \sqrt{\frac{2}{\pi}}|X_i|$. Note that,

$$
\begin{aligned}
V\left[\sqrt{\frac{\pi}{2}}|X_1|\right] &= \frac{\pi}{2}\{E(X_1^2) - [E(|X_1|)]^2\} \\
&= \frac{\pi}{2}\left[\theta\left(1 - \frac{2}{\pi}\right)\right] = \theta\left[\frac{\pi}{2} - 1\right].
\end{aligned}
$$

By independence,

$$V(Y) = \theta\left[\frac{\pi}{2} - 1\right]\frac{1}{n}. \tag{6.0.1}$$

To finish, we need the efficiency of the parameter $\sqrt{\theta}$. For convenience, let $\beta = \sqrt{\theta}$. Then

$$\log f(x;\beta) = -\log\sqrt{2\pi} - \log\beta - \frac{1}{2}\frac{x^2}{\beta^2}.$$

The second partial of this expression is,

$$\frac{\partial^2 \log f(x;\beta)}{\partial \beta^2} = \frac{1}{\beta^2} - 3\frac{x^2}{\beta^4}.$$

Hence, using $\sqrt{\theta} = \beta$,

$$I(\sqrt{\theta}) = -E\left[\frac{1}{\theta} - 3\frac{X^2}{\theta^2}\right] = \frac{2}{\theta}. \tag{6.0.2}$$

Thus by (6.0.1) and (6.0.2) we have

$$e(Y) = \frac{\theta/2n}{\theta[(\pi/2) - 1]/n} = \frac{1}{\pi - 2}.$$

6.2.14 For Part (a), recall that $(n-1)S^2/\theta$ has $\chi^2(n-1)$ distribution. Hence,

$$V\left[\frac{(n-1)S^2}{\theta}\right] = 2(n-1).$$

So $V(S^2) = 2\theta^2/(n-1)$. Also, by Problem (6.28), $I(\theta) = (2\theta^2)^{-1}$. Thus, the efficiency of S^2 is $(n-1)/n$.
