STAT 517 HW8 Solution:

8.3.5 Say $H_0 : \theta = \theta_0$ and $H_1 : \theta = \theta_1$. The likelihood ration test is

$$\lambda = \frac{f(x_1;\theta_0)f(x_2;\theta_0)\cdots f(x_n;\theta_0)}{\max[f(x_1;\theta_i)f(x_2;\theta_i)\cdots f(x_n;\theta_i), i = 0, 1]} \le k$$

If the maximum in the denominator occurs when $i = 0$, the $\lambda = 1$ and we do not reject. If that maximum occurs when $i = 1$, then

$$\lambda = \frac{f(x_1;\theta_0)\cdots f(x_n;\theta_0)}{f(x_1;\theta_1)\cdots f(x_n;\theta_1)} \le k$$

which is the critical region given by the Neyman-Pearson theorem.

8.3.6

$$\lambda = \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum(x_i - \theta')^2/2\right]}{\left(\frac{1}{\sqrt{2\pi}}\right)^n \exp\left[-\sum(x_i - \bar{x})^2/2\right]} \le k$$

is equivalent to

$$\exp\left\{\left[-\sum(x_i - \bar{x})^2 - n(\bar{x} - \theta')^2 + \sum(x_i - \bar{x})^2\right]/2\right\} \le c_1$$

and thus

$$n(\bar{x} - \theta) \ge c_2 \text{ and } |\bar{x} - \theta'| \ge c_3.$$

8.3.9 Since $\sum |x_i - \theta|$ is minimized when $\hat{\theta} = \text{median}(X_i) = Y_3$, the likelihood ratio test is

$$\lambda = \frac{\left(\frac{1}{2}\right)^5 \exp\left[-\sum |x_i - \theta_0|\right]}{\left(\frac{1}{2}\right)^5 \exp\left[-\sum |x_i - y_3|\right]} \le k.$$

This is the equivalent to

$$\exp\left[-\sum |x_i - y_3| - 5|y_3 - \theta_0| + \sum |x_i - y_3|\right] \le k$$

and

$$|y_3 - \theta_0| \ge c.$$

8.3.11 The likelihood function for this problem is

$$L(\theta) = \theta^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\theta-1}.$$

(a) For $\theta_1 < \theta_2$, the ratio of the likelihoods is

$$\frac{L(\theta_1)}{L(\theta_2)} = \left(\frac{\theta_1}{\theta_2}\right)^n \left[\prod_{i=1}^{n}(1-x_i)\right]^{\theta_1-\theta_2}.$$

This has decreasing monotone-likelihood-ratio in the statistic $\prod_{i=1}^n(1-x_i)$. Hence, the UMP test, rejects H_0 if $\prod_{i=1}^n(1-x_i) \geq c$.

(b) Taking the partial derivative of the log of the likelihood, yields the mle estimator:

$$\widehat{\theta} = \frac{n}{-\log\prod_{i=1}^n(1-x_i)}.$$

The likelihood ratio test statistic is

$$\Lambda = \frac{1}{\widehat{\theta}^n(\prod_{i=1}^n(1-x_i))^{\widehat{\theta}-1}}.$$

8.4.2

$$\begin{aligned}
\frac{0.2}{0.9} \approx k_0 \quad &< \quad \frac{(0.02)^{\sum x_i}e^{-n(0.02)}}{(0.07)^{\sum x_i}e^{-n(0.07)}} < k_1 \approx \frac{0.8}{0.1} \\
\frac{2}{9} \quad &< \quad \left(\frac{2}{7}\right)^{\sum x_i} e^{(0.05)n} < 8 \\
c_1(n) \quad &= \quad \frac{\log(2/9)-(0.05)n}{\log(2/7)} > \sum x_i > \frac{\log 8-(0.05)n}{\log(2/7)} = c_0(n).
\end{aligned}$$

8.4.4

$$\begin{aligned}
\frac{0.02}{0.98} \quad &< \quad \frac{(0.01)^{\sum x_i}(0.99)^{100n-\sum x_i}}{(0.05)^{\sum x_i}(0.95)^{100n-\sum x_i}} < \frac{0.98}{0.02} \\
-\log 49 \quad &< \quad (\sum x_i)\log\left(\frac{19}{99}\right) + 100n\log\left(\frac{99}{95}\right) < \log 49 \\
\frac{[-100\log(99/95)]n - \log 49}{\log(19/99)} \quad &> \quad \sum x_i > \frac{-100\log(99/95)] + \log 49}{\log(19/99)}
\end{aligned}$$

or, equivalently,

$$\frac{\log 49}{\log(99/19)} > \sum\left[x_i - 100\frac{\log(99/95)}{\log(99/19)}\right] > \frac{-\log 49}{\log(99/19)}.$$