HW6 Solution

7.8.2

(a)

$$\prod_{i=1}^{n}\left(\frac{1}{2\theta}\right)I_{[-\theta,\theta]}(x_i)=\left(\frac{1}{2\theta}\right)^n I_{[-\theta,y_n]}(y_1)I_{[y_1,\theta]}(y_n);$$

by the factorization theorem, the pair $(Y_1, Y_n)$ is sufficient for $\theta$.

(b) $L = \left(\frac{1}{2\theta}\right)^n$, provided $-\theta \le y_1$ and $y_n \le \theta$. That is, $-y_1 \le \theta$ and $y_n \le \theta$. We want to make $\theta$ as small as possible and satisfy these two restrictions; hence $\hat{\theta} = \max(-Y_1, Y_n)$.

(c) It is easy to show from the joint pdf $Y_1$ and $Y_n$ that the pdf of $\hat{\theta}$ is $g(z;\theta) = nz^{n-1}/\theta^n, 0 \le z \le \theta$, zero elsewhere. Hence

$$L/g(z;\theta)=\frac{1}{2^n(nz^{n-1})},\quad -z=-\hat{\theta}\le x_i\le\hat{\theta}=z,$$

which is free of $\theta$.

7.8.5 it will be easy to check scale invariant property.

7.8.4 For illustration $Y_{n-2} - Y_3, \min(-Y_1, Y_n)/\max(Y_1, Y_n)$ and $(Y_2 - Y_1)/\sum(Y_i - Y_1)$, respectively.

7.9.3 From previous results (Chapter 3), we know that $Z$ and $Y$ have a bivariate normal distribution. Thus they are independent if and only if their covariance is equal to zero; that is

$$\sum_{i=1}^{n} a_i\sigma^2 = 0 \text{ or, equivalently, } \sum_{i=1}^{n} a_i = 0.$$

If $\sum a_i = 0$, note that $\sum a_i X_i$ is location-invariant because $\sum a_i(x_i + d) = \sum a_i x_i$.

7.9.5 Of course, $R$ is a scale-invariant statistic, and thus $R$ and the complete sufficient statistic $\sum_1^n Y_i$ for $\theta$ are independent. Since $M_1(t) = E[\exp(tnY_1)] = (1-\theta t)^{-1}$ for $t < 1/\theta$, and $M_2(t) = E[\exp(t\sum_1^n Y_i)] = (1-\theta t)^{-n}$ we have

$$M_1^{(k)}(0) = \theta^k\Gamma(k+1) \text{ and } M_2^{(k)}(0) = \theta^k\Gamma(n+k)/\Gamma(n).$$

According to the result of Exercise 7.9.4 we now have $E(R^k) = M_1^{(k)}(0)/M_2^{(k)}(0) = \Gamma(k+1)\Gamma(n)/\Gamma(n+k)$. These are the moments of a beta distribution with $\alpha = 1$ and $\beta = n-1$.

7.9.7 The two ratios are location- and scale-invariant statistics and thus are independent of the joint complete and sufficient statistic for the location and scale parameters, namely $\bar{X}$ and $S^2$.

7.9.9

(a) Here $R$ is a scale-invariant statistic and hence independent of the complete and sufficient statistic, $\sum X_i^2$, for $\theta$, the scale parameter.

(b) While the numerator, divided by $\theta$, is $\chi^2(2)$ and the denominator, divided by $\theta$, is $\chi^2(5)$, they are not independent and hence 5R/2 does not have an F-distribution.

(c) It is easy to get the moment of the numerator and denominator and thus the quotient of the corresponding moments to show that $R$ has a beta distribution.