7.5.3.

K(x) = log(x), p(θ) = $\theta - 1$, q(θ) = log(θ)

So log(x1) + …+ log(xn) is a complete sufficient statistic for θ.

So the geometric mean is also a complete sufficient statistic for θ.

7.5.5

$$\frac{d}{d\theta}\int_a^b \exp\big(p(\theta)K(x) + S(x) + q(\theta)\big)\,dx = \frac{d}{d\theta}1 = 0$$

$$= \int_a^b \frac{d}{d\theta}\exp\big(p(\theta)K(x) + S(x) + q(\theta)\big)\,dx$$

So $p'(\theta)E\big(K(X)\big) = -q'(\theta)$.

For variance, take second derivative and exchange integration and d/dθ.

7.5.6

$$\begin{aligned} E\left[e^{tK(X)}\right] &= \int_a^b \exp\{(t+\theta)K(x) + S(x) + q(\theta)\}\,dx \\ &= \exp\{q(\theta) - q(\theta - t)\}\int_a^b \exp\{(t+\theta)K(x) + S(x) + q(\theta+t)\}\,dx. \end{aligned}$$

However the integral equals one since the integrand can be treated as a pdf, provided $\gamma < \theta + t < \delta$.

7.6.5 For part (a), since $Y = \sum_{i=1}^n X_i$, we have

$$\begin{aligned} P[X_1 \le 1 | Y = y] &= P[X_1 = 0|Y = y] + P[X_1 = 1|Y = y] \\ &= \frac{P[\{X_1 = 0\} \cap \{\sum_{i=2}^n X_i = y\}]}{P(Y = y)} \\ &\quad + \frac{P[\{X_1 = 1\} \cap \{\sum_{i=2}^n X_i = y - 1\}]}{P(Y = y)} \\ &= \frac{e^{-\theta}e^{-(n-1)\theta}[(n-1)\theta]^y/y!}{e^{-n\theta}(n\theta)^y/y!} \\ &\quad + \frac{e^{-\theta}\theta e^{-(n-1)\theta}[(n-1)\theta]^{y-1}/(y-1)!}{e^{-n\theta}(n\theta)^y/y!} \\ &= \left(\frac{n-1}{n}\right)^y + \frac{y}{n-1}\left(\frac{n-1}{n}\right)^y \\ &= \left(\frac{n-1}{n}\right)^y\left(1 + \frac{y}{n-1}\right). \end{aligned}$$

Hence, the statistic $\left(\frac{n-1}{n}\right)^Y\left(1 + \frac{Y}{n-1}\right)$ is the MVUE of $(1+\theta)e^{-\theta}$.

$$\begin{aligned} h(z|y) &= \frac{(n-1)(y-z)^{n-2}}{y^{n-1}}, \ 0 < z < y. \\ E\left[I_{(0,2)}(Z)|y\right] &= \int_0^\infty \left\{\left[I_{(0,2)}(z)\right](n-1)(y-z)^{n-2}/y^{n-1}\right\} dy \\ &= 1 - \left(\frac{y-2}{y}\right)^{n-1} = 1-(1-2/y)^{n-1}. \end{aligned}$$

That is, the MVUE estimator is

$$\left(1 - \frac{2/\bar{X}}{n}\right)^{n-1}.$$

Of course, this is approximately equals to the mle when n is large.

7.6.8 $P(X \le 2) = \int_0^2 (1/\theta)e^{-x/\theta}\, dx = 1 - e^{-2/\theta}$. Since $\bar{X} = Y/n$, where $Y = \sum X_i$, is the mle of θ, then the mle of that probability is $1 - e^{-2/\bar{X}}$. Since $I_{(0,2)}(X_1)$ is an unbiased estimator of $P(X \le 2)$, let us find the joint pdf of $Z = X_1$ and Y by first letting $V = X_1 + X_2, U = X_1 + X_2 + X_3 + \ldots$. The Jacobian is one; then we integrate out those other variables obtaining

$$g(z, y; \theta) = \frac{(y-z)^{n-2}e^{y/\theta}}{(n-2)!\theta^n}, \ 0 < z < y < \infty.$$

Since the pdf of Y is

$$g_2(y;\theta) = \frac{y^{n-1}e^{-y/\theta}}{(n-1)!\theta^n}, \ 0 < y < \infty,$$

we have that the conditional pdf of Z, given $Y = y$, is

7.7.3

$$\begin{aligned} f(x,y) &= \exp\left\{\left[\frac{-1}{2(1-\rho^2)\sigma_1^2}\right]x^2 + \left[\frac{-1}{2(1-\rho^2)\sigma_2^2}\right]y^2 + \left[\frac{\rho}{(1-\rho^2)\sigma_1\sigma_2}\right]xy \right. \\ &\quad + \left[\frac{\mu_1}{(1-\rho)\sigma_1^2} - \frac{\rho\mu_2}{(1-\rho^2)\sigma_1\sigma_2}\right]x + \left[\frac{\mu_2}{(1-\rho^2)\sigma_2^2} - \frac{\rho\mu_1}{(1-\rho^2)\sigma_1\sigma_2}\right]y \\ &\quad \left. + q(\mu_1, \mu_2, \sigma_1, \sigma_2, \rho)\right\} \end{aligned}$$

Hence $\sum X_i^2, \sum Y_i^2, \sum X_iY_i, \sum X_i, \sum Y_i$ are joint complete sufficient statistics. Of course, the other five provide a one-to-one transformation with these five; so they are also joint complete and sufficient statistic.

7.7.9 Part (a): Consider the following function of the sufficient and complete statistics

$$\begin{aligned} \mathbf{W} &= \sum_{i=1}^{n} (\mathbf{X}_i - \overline{\mathbf{X}})(\mathbf{X}_i - \overline{\mathbf{X}})' \\ &= \sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i' - n\overline{\mathbf{X}}\overline{\mathbf{X}}'. \end{aligned}$$

Recall that the variance-covariance matrix of a random vector $\mathbf{Z}$ can be expressed as

$$\operatorname{cov}(\mathbf{Z}) = E[\mathbf{Z}\mathbf{Z}'] - E[\mathbf{Z}]E[\mathbf{Z}]'.$$

In the notation of the example, we have

$$E[\sum_{i=1}^{n} \mathbf{X}_i \mathbf{X}_i'] = \sum_{i=1}^{n} E[\mathbf{X}_i \mathbf{X}_i'] = n\boldsymbol{\Sigma} + n\boldsymbol{\mu}\boldsymbol{\mu}'.$$

But the random vector $\overline{\mathbf{X}}$ has mean $\boldsymbol{\mu}$ and variance-covariance matrix $n^{-1}\boldsymbol{\Sigma}$. Hence,

$$E[\overline{\mathbf{X}}\overline{\mathbf{X}}'] = n^{-1}\boldsymbol{\Sigma} + \boldsymbol{\mu}\boldsymbol{\mu}'.$$

Putting these last two results together

$$E[\mathbf{W}] = (n-1)\boldsymbol{\Sigma},$$

i.e., $\mathbf{S} = (n-1)^{-1}\mathbf{W}$ is an unbiased estimator of $\boldsymbol{\Sigma}$. Thus the $(i, j)th$ entry of $\mathbf{S}$ is the MVUE of σ_{ij}.

7.7.12 The order statistics are sufficient and complete and $\overline{X}$ is a function of them. Further, $\overline{X}$ is unbiased. Hence, $\overline{X}$ is the MVUE of μ.