STAT 517 HW4:

7.2.3 $f(x;\theta) = Q(\theta)M(x)I_{(0,\theta)}(x)$. Therefore

$$\prod_{i=1}^{n} Q(\theta)M(x_i)I_{(0,\theta)}(x_i) = \{[Q(\theta)]^n I_{(0,\theta)}[\max(x_i)]\}\left\{\prod_{i=1}^{n} M(x_i)\right\},$$

because $\prod I_{(0,\theta)}(x_i) = \prod I_{(0,\theta)}[\max(x_i)]$. According to the factorization theorem, $Y = \max(X_i)$ is a sufficient statistic for θ.

7.2.5

$$\frac{f(x1;\theta)f(x2;\theta)\ldots f(xn;\theta)}{f_Y(u(x1,\ldots,xn);\theta)} = \frac{\Gamma(\mathrm{n})}{(n\bar{x})^{n-1}}H(x1,\ldots,xn)$$

which does not depend on θ, so $n\bar{x}$ is sufficient.

7.2.8

$$\prod_{i=1}^{n} \frac{\Gamma(2\theta)}{[\Gamma(\theta)]^2}[x_i(1-x_i)]^{\theta-1} = \left\{\frac{[\Gamma(2\theta)]^n}{[\Gamma(\theta)]^{2n}}\left[\prod x_i(1-x_i)\right]^{\theta-1}\right\}(1).$$

Thus $Y = \prod[X_i(1-X_i)]$ is a sufficient statistic for θ.

7.3.1

(1)

$(\frac{1}{2\pi\theta})^{n/2}\exp(\sum_1^n X_i^2/2\theta^2)$ is the loglikelihood function.

Take derivative. $\sum_1^n X_i^2/n$

(2)loglikelihood $c - \log n\theta$ So max {Xi} is the estimator.

7.3.5 For illustration, in Exercise 7.2.1, $Y = \sum X_i^2$ is a sufficient statistic, and

$$E(Y) = \sum E(X_i^2) = \sum \theta = n\theta$$

Thus $Y/n = \sum X_i^2/n$ is an unbiased estimator.

7.3.6 It suffices to find the conditional distribution of X_1 given $\sum_{i=1}^n X_i = x$. Assuming $x \geq x_1$ (otherwise the following probability is 0) we have

$$\begin{aligned} P[X_1 = x_1 | \sum_{i=1}^n X_i = x] &= \frac{P[X_1 = x_1, \sum_{i=1}^n X_i = x]}{P[\sum_{i=1}^n X_i = x]} \\ &= \frac{P[X_1 = x_1, \sum_{i=2}^n X_i = x - x_1]}{P[\sum_{i=1}^n X_i = x]} \\ &= \frac{e^{-\theta}\frac{\theta^{x_1}}{x_1!} e^{-(n-1)\theta}\frac{[(n-1)\theta]^{x-x_1}}{(x-x_1)!}}{e^{-n\theta}\frac{[n\theta]^x}{x!}} \\ &= \binom{x}{x_1}\left(\frac{1}{n}\right)^{x_1}\left(1-\frac{1}{n}\right)^{x-x_1}. \end{aligned}$$

Thus, the conditional distribution is binomial and $E[X_1 | \sum_{i=1}^n X_i = x] = x/n$. By linearity of conditional expectation it follows that

$$E[X_1 + 2X_2 + 3X_3 | \sum_{i=1}^n X_i = x] = (6x)/n = 6\overline{x}.$$

7.4.2 In each case $E(X) = 0$ for all $\theta > 0$.

7.4.3 A generalization of 7.4.1. Since $E[\sum X_i] = n\theta$, $\sum X_i/n$ is the unbiased minimum variance estimator.