STAT 517

Inference about normal models: quadratic forms

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Definition

- A polynomial of degree 2 in n variables is a **quadratic form**
- Examples: $X_1^2 + X_1X_2 + X_2^2$ is a quadratic form in X_1 and X_2
- $X_1^2 + X_2^2 + X_3^2 - 2X_1X_2$ is a quadratic form in X_1, X_2, X_3
- Caution: $(X_1 - 1)^2 + (X_2 - 2)^2$ is a quadratic form in $X_1 - 1$ and $X_2 - 1$ but *NOT* in X_1 and X_2

Why do we care

- Consider the sample variance S^2 of a random sample $X_1, X_2, \ldots, X_n$
- $(n-1)S^2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{n-1}{n}(X_1^2 + \cdots + X_n^2) - \frac{2}{n}(X_1X_2 + \cdots + X_1X_n + \cdots + X_{n-1}X_n)$
- Thus, $(n-1)S^2$ is a quadratic form in $X_1, \ldots, X_n$
- Recall that if $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$, $(n-1)S^2/\sigma^2 \sim \chi^2_{n-1}$ regardless of μ
- Tests of certain hypotheses result in quadratic form valued statistics, e.g. $\sum_{i=1}^{n} X_i^2$

Hogg-Craig theorem

- Let $Q = Q_1 + Q_2 + \cdots + Q_k$ where $Q, Q_1, \cdots, Q_k$ are $k+1$ random variables that are quadratic forms in n independent random variables
- n variables $X_1, \ldots, X_n \sim N(\mu, \sigma^2)$
- Let $Q/\sigma^2, Q_1/\sigma^2, \ldots, Q_{k-1}/\sigma^2$ be $\chi^2_r, \chi^2_{r_1}, \ldots, \chi^2_{r_{k-1}}$ respectively
- Then
  1. $Q_1, \ldots, Q_k$ are independent
  2. $Q_k/\sigma^2 \sim \chi^2$ with $r - (r_1 + \cdots + r_{k-1}) = r_k$ df

Example

- Define
  $X_{11}, \ldots, X_{1b}, X_{21}, \ldots, X_{2b}, \ldots, X_{a1}, \ldots, X_{ab} \sim N(\mu, \sigma^2)$
- The total sample size is $n = ab$; by assumption, all are independent
- Define the following $a + b + 1$ statistics:

$$\bar{X}_{..} = \frac{1}{ab} \sum_{i=1}^{a} \sum_{j=1}^{b} X_{ij}, \bar{X}_{i.} = \frac{1}{b} \sum_{j=1}^{b} X_{ij}$$

$$\bar{X}_{.j} = \frac{1}{a} \sum_{i=1}^{a} X_{ij}$$

Example

- Verify that $(ab-1)S^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij}-\bar{X}_{i.})^2 + b\sum_{i=1}^{a}(\bar{X}_{i.}-\bar{X}_{..})^2$
- For brevity, $Q = Q_1 + Q_2$
- Immediately, $Q/\sigma^2 = (ab-1)S^2/\sigma^2 \sim \chi^2_{ab-1}$
- Moreover, $Q_1/\sigma^2 \sim \chi^2_{a(b-1)}$
- By Hogg-Craig's theorem, $Q_2/\sigma^2 \sim \chi^2_{a-1}$ because $ab-1-a(b-1) = a-1$

Example II

- Similarly, obtain

$$(ab-1)S^2 = \sum_{i=1}^{a}\sum_{j=1}^{b}(X_{ij} - \bar{X}_{.j})^2 + a\sum_{j=1}^{b}(\bar{X}_{.j} - \bar{X}_{..})^2$$

- For brevity, $Q = Q_3 + Q_4$
- Show that $Q_3/\sigma^2 \sim \chi^2_{b(a-1)}$; therefore, Q_3 and Q_4 are independent...and $Q_4/\sigma^2 \sim \chi^2_{b-1}$
- Why? Because $ab - 1 - b(a-1) = b - 1$