STAT 517

Sequential probability ratio test

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March 29, 2016

Set-up

- $X_1, \ldots, X_n \sim f(x;\theta)$, sample size is a random $N \in \{1,2,3,\ldots\}$
- $H_0 : \theta = \theta^{'}$ vs. $H_1 : \theta = \theta^{''}$; define $0 < k_0 < k_1$
- Denote the likelihood $L(\theta; n)$ to symbolize its dependence on n
- Compute consecutively $\frac{L(\theta^{'};1)}{L(\theta^{"};1)}$, $\frac{L(\theta^{'};2)}{L(\theta^{"};2)}$ etc.

Set-up

- Reject $H_0 : \theta = \theta^{'}$ if and only if there exists a positive integer n s.t. $\mathbf{x}_n = (x_1, \ldots, x_n)$ belongs to the set

$$C_n = \left\{ \mathbf{x}_n : k_0 < \frac{L(\theta^{'}; j)}{L(\theta^{''}; j)} < k_1, j = 1, \ldots, n-1, \frac{L(\theta^{'}; n)}{L(\theta^{''}; n)} \le k_0 \right\}$$

- Fail to reject H_0 if and only ifthere exists a positive integer n s.t. $\mathbf{x}_n = (x_1, \ldots, x_n)$ belongs to the set

$$B_n = \left\{ \mathbf{x}_n : k_0 < \frac{L(\theta^{'}; j)}{L(\theta^{''}; j)} < k_1, j = 1, \ldots, n-1, \frac{L(\theta^{'}; n)}{L(\theta^{''}; n)} \ge k_1 \right\}$$

Example

- Often, the needed inequality is expressed in the form

$$c_0(n) < u(x_1, \dots, x_n) < c_1(n)$$

where $c_0(n)$ and $c_1(n)$ depend on k_0, k_1, θ', $\theta''(n)$ and n
- Take $X \sim b(1, \theta)$, $H_0 : \theta = \frac{1}{3}$ and $H_1 : \theta = \frac{2}{3}$, check

$$\frac{L(1/3, n)}{L(2/3, n)} = 2^{n - 2\sum_{i=1}^{n} x_i}$$

- Transform $k_0 < \frac{L(1/3,n)}{L(2/3,n)} < k_1$ into

$$c_0(n) = \frac{n}{2} - \frac{1}{2}\log_2 k_1 < \sum x_i < \frac{n}{2} - \frac{1}{2}\log_2 k_0 = c_1(n)$$

- Reject H_0 if $c_1(n) \leq \sum x_i$, fail to reject H_0 if $c_0(n) \geq \sum x_i$

Power function computation

- Probability of Type I error is α, power is $1-\beta$
- Clearly, $\alpha = \sum_{n=1}^{\infty} \int_{C_n} L(\theta^{'}, n)$ and $1-\beta = \sum_{n=1}^{\infty} \int_{C_n} L(\theta^{''}, n)$
- The procedure terminates with probability 1, so $1-\alpha = \sum_{n=1}^{\infty} \int_{B_n} L(\theta^{'}, n)$ and $\beta = \sum_{n=1}^{\infty} \int_{B_n} L(\theta^{''}, n)$
- Simple algebra results in $\frac{\alpha}{1-\beta} \le k_0$, $k_1 \le \frac{1-\alpha}{\beta}$
- Select prior α_a and β_a; typically, 0.01, 0.05 or 0.10 s.t. $k_0 = \frac{\alpha_a}{1-\beta_a}$ and $k_1 = \frac{1-\alpha_a}{\beta_a}$
- Verify that $\alpha + \beta \le \alpha_1 + \beta_a$ - upper bound on the sum of two errors
- Clearly, $\alpha \le \frac{\alpha_a}{1-\beta_a}$ and $\beta \le \frac{\beta_a}{1-\alpha_a}$
- In practice, it is common to approximate the power at $\theta = \theta^{'}$ with α and at $\theta = \theta^{''}$ with $1-\beta_a$

Example

- Take $X \sim N(\theta, 100)$; test $H_0 : \theta = 75$ vs. $H_1 : \theta = 78$, $\alpha = \beta = 0.10$
- Find that $k_0 = \frac{0.10}{1-0.10} = \frac{1}{9}$ and $k_1 = 9$
- Simple algebra gives

$$c_0(n) = \frac{153}{2}n - \frac{100}{3}\log 9 < \sum x_i < \frac{153}{2}n + \frac{100}{3}\log 9 = c_1(n)$$

- Reject H_0 if $\sum x_i \geq c_1(n)$ and fail to reject if $\sum x_i \leq c_0(n)$