STAT 517: Sufficiency

Uniformly most powerful tests (UMP) and likelihood ratio tests

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Typically, it is important to handle the case where the alternative hypothesis may be a **composite** one.

It is desirable to have the best critical region for testing $H_0$ against each simple hypothesis in $H_1$.

The critical region $C$ is **uniformly most powerful (UMP)** of size $\alpha$ against $H_1$ if it is so against *each* simple hypothesis in $H_1$.

A test defined by such a regions is a **uniformly most powerful (UMP)** test.

Such a test does not always exist.
Example

- $X_1, \ldots, X_n \sim N(0, \theta)$
- $H_0: \theta = \theta'$ vs. $H_1: \theta > \theta'$
- Note that the likelihood is

$$L(\theta; x_1, \ldots, x_n) = \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp \left\{-\frac{1}{2\theta} \sum_{i=1}^{n} x_i^2\right\}$$

- Verify that the critical region $C : \{x : \sum_{i=1}^{n} x_i^2 \geq c\}$ is the best critical region for such a test
- To find the critical value of $c$, recall that $\sum_{i=1}^{n} X_i^2 / \theta' \sim \chi_n^2$
Example

- If $X_1, \ldots, X_n \sim N(\theta, 1)$ and we test $H_0 : \theta = \theta'$ vs. $H_1 : \theta \neq \theta'$ there is no UMP

- Verify that there are two rejection regions:

  \[
  \sum_{i=1}^{n} x_i \geq \frac{n}{2}(\theta'' - \theta') - \frac{\log k}{\theta'' - \theta'}
  \]

  if $\theta'' > \theta'$ and

  \[
  \sum_{i=1}^{n} x_i \leq \frac{n}{2}(\theta'' - \theta') - \frac{\log k}{\theta'' - \theta'}
  \]

  if $\theta'' < \theta'$

- If the alternative $H_1 : \theta > \theta'$ or $H_1 : \theta < \theta'$, the UMP does exist
Example: a UMP test for Poisson means

- Take $X_1, \ldots, X_n \sim P(\theta)$, $H_0 : \theta = 0.1$ vs. $H_1 : \theta > 0.1$
- The alternative $\theta'' > 0.1$ the likelihood ratio can be represented as

$$\left(\frac{0.1}{\theta''}\right)^{\sum x_i} e^{-10(1-\theta'')} \leq k$$

- The equivalent form is

$$\sum_{i=1}^{n} x_i \geq \frac{\log k + 10 - 10\theta''}{\log 0.1 - \log \theta''}$$

- The best critical region has the form $\sum_{i=1}^{n} x_i \geq c$ for a constant $c$
A useful remark

▶ If $X_1, \ldots, X_n \sim f(x; \theta)$, $Y = u(X_1, \ldots, X_n)$ is a sufficient statistic for $\theta$, the ratio

$$\frac{L(\theta'; x_1, \ldots, x_n)}{L(\theta''; x_1, \ldots, x_n)} = \frac{k_1[u(x_1, \ldots, x_n); \theta']}{k_1[u(x_1, \ldots, x_n); \theta'']}$$

by factorization theorem

▶ If a best test or a UMP test is desired, only functions of sufficient statistics should be considered
General form of uniformly most powerful tests

- Take a general one-sided hypothesis: $H_0 : \theta \leq \theta'$ vs. $H_1 : \theta > \theta'$
- The level of the test is

$$\max_{\theta \leq \theta'} \gamma(\theta)$$

the maximum probability of Type I error
- The general likelihood function is

$$L(\theta, x) = \prod_{i=1}^{n} f(x_i; \theta)$$
The likelihood \( L(\theta; x) \) has the **monotone likelihood ratio** in statistic \( y = u(x) \) for \( \theta_1 < \theta_2 \) if
\[
\frac{L(\theta_1; x)}{L(\theta_2; x)}
\]
is a monotone function of \( y = u(x) \).

If our likelihood \( L(\theta; x) \) has a monotone decreasing likelihood ratio in \( y = u(x) \), the test “Reject \( H_0 \) if \( Y \geq c_Y \)” is the UMP.

Here, \( \alpha = P_{\theta'}[Y \geq c_Y] \).
Example: Bernoulli UMP

- Take $X_1, \ldots, X_n \sim b(1, \theta), 0 < \theta < 1$.

- For $\theta' < \theta''$, the likelihood ratio is

\[
\frac{\theta' (1 - \theta'')}{\theta'' (1 - \theta')} \sum x_i \left( \frac{1 - \theta'}{1 - \theta''} \right)^n
\]

- $\frac{\theta' (1 - \theta'')}{\theta'' (1 - \theta')} < 1$ and so the ratio is a decreasing function of $y = \sum x_i$.

- We have the MLR in $Y = \sum X_i$.

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Example: Bernoulli UMP

The conclusion: the UMP level $\alpha$ decision rule to test $H_0$ vs. $H_1$ is: reject $H_0$ when

$$Y = \sum_{i=1}^{n} X_i \geq c$$

Here, $c$ is s. t. $\alpha = P_{\theta'}(Y \geq c)$
General UMP for regular exponential families

- The regular family pdf is
  \[ f(x; \theta) = \exp[p(\theta)K(x) + H(x) + q(\theta)] \]

- The likelihood ratio is
  \[ \frac{L(\theta')}{L(\theta'')} = \exp \left\{ [p(\theta') - p(\theta'')] \sum_{i=1}^{n} K(x_i) + n[q(\theta') - q(\theta'')] \right\} \]

- If \( p(\theta) \) is an increasing function of \( \theta \), the ratio is a decreasing function of \( y = \sum_{i=1}^{n} K(x_i) \)

- For \( H_0 : \theta \leq \theta'' \) vs. \( H_1 : \theta > \theta' \), the UMP level \( \alpha \) decision rule is to reject \( H_0 \) if
  \[ Y = \sum_{i=1}^{n} K(X_i) \geq c \]

- Here \( c \) is s.t. \( \alpha = P_{\theta'}(Y \geq c) \)
Now, let $H_0 : \theta \geq \theta'$ vs. $H_1 : \theta < \theta'$

The UMP level $\alpha$ decision rule is (again, if $p(\theta)$ is an increasing function) to reject $H_0$ if

$$Y = \sum_{i=1}^{n} K(X_i) \leq c$$

Here, $c$ is s. t. $\alpha = P_{\theta'}(Y \leq c)$
The case of a simple null hypothesis

- If $H_0 : \theta = \theta'$ and $H_1 : \theta > \theta'$, $\sum_{i=1}^{n} K(x_i) \geq c$ is a UMP critical region
- All of the regions considered earlier: $\sum_{i=1}^{n} x_i^2 \geq c$, $\sum_{i=1}^{n} x_i \geq c$ etc. are UMP for testing $H_0$ vs $H_1$
General Likelihood Ratio Tests: a possible setting

- What if $H_0 : \theta \in \omega$ and $H_1 : \theta \in \Omega \cap \omega'$?
- Recall testing the normal mean with unknown variance
- $X_1, \ldots, X_n \sim N(\theta_1, \theta_2), \theta = (\theta_1, \theta_2)$
- $\Omega = \{\theta : -\infty < \theta_1 < \infty, \theta_2 > 0\}$, $\omega = \{\theta : \theta_1 = \theta'_1, \theta_2 > 0\}$
Possibilities

- If $\theta_2$ is known, we found earlier that there is no UMP test.
- It is possible to construct a theory of best tests if unbiasedness is required.
- A specific critical region will be

$$C_2 = \left\{ |\bar{X} - \theta'_1| > \sqrt{\frac{\theta_2}{n}} z_{\alpha/2} \right\}$$

- $C_2$ is unbiased and provides a UMP unbiased test of level $\alpha$. 
Comparison of two normal means

- $X_1, \ldots, X_n \sim N(\theta_1, \theta_3)$, $Y_1, \ldots, Y_n \sim N(\theta_2, \theta_3)$
- All of the parameters are unknown
- $H_0 : \theta_1 = \theta_2$
- Then, $\Omega = \{(\theta_1, \theta_2, \theta_3) : -\infty < \theta_1, \theta_2 < \infty, \theta_3 > 0\}$
- $\omega = \{(\theta_1, \theta_2, \theta_3) : \theta_1 = \theta_2, \theta_3 > 0\}$
MLE

\[
L(\omega) = \left( \frac{1}{2\pi\theta_3} \right)^{(n+m)/2} \times 
\exp \left\{ -\frac{1}{2\theta_3} \left[ \sum_{i=1}^{n} (x_i - \theta_1)^2 + \sum_{i=1}^{m} (y_i - \theta_1)^2 \right] \right\}
\]

\[
L(\Omega) = \left( \frac{1}{2\pi\theta_3} \right)^{(n+m)/2} \times 
\exp \left\{ -\frac{1}{2\theta_3} \left[ \sum_{i=1}^{n} (x_i - \theta_1)^2 + \sum_{i=1}^{m} (y_i - \theta_2)^2 \right] \right\}
\]
Let $\omega$ and $\omega'$ be the MLE solutions for the variance parameter $\theta_3$

The likelihood ratio is

$$\Lambda = \left( \frac{\omega'}{\omega} \right)^{(n+m)/2}$$

We can transform the above to find that

$$\Lambda^2/(n+m) = \frac{n + m - 2}{(n + m - 2) + T^2}$$

where $T \sim t_{n+m-2}$ under the null hypothesis

The significance level of this test is

$$\alpha = P_{H_0}[\Lambda(\mathbf{X}_1, \ldots, \mathbf{X}_n, Y_1, \ldots, Y_n) \leq \lambda_0]$$

The above inequality is equivalent to $|T| \geq c$ so

$$\alpha = P(|T| \geq c; H_0)$$
Non-central t distribution

- Let $W \sim N(\delta, 1)$, $V \sim \chi^2_r$, $W$ and $V$ are independent
- The quotient
  \[
  T = \frac{W}{\sqrt{V/r}}
  \]
  has a non-central $t$ distribution with $r$ df and noncentrality parameter $\delta$
- The usual $t$ distribution is a special case with $\delta = 0$
*t*-test for the mean and its power

- Let $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$; test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$

- Recall

\[
t(X_1, \ldots, X_n) = \frac{\sqrt{n\bar{X}}}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 / (n - 1)}}
\]

- Clearly,

\[
t(X_1, \ldots, X_n) = \frac{\sqrt{n\bar{X}} / \sigma}{\sqrt{\sum_{i=1}^n (X_i - \bar{X})^2 / [\sigma^2(n - 1)]}} = \frac{W_1}{\sqrt{V_1 / (n - 1)}}
\]

with $W_1 = \sqrt{n\bar{X}} / \sigma$, $V_1 = \sum_{i=1}^n (X_i - \bar{X})^2 / \sigma^2$
$W_1 \sim N(\sqrt{n}\theta_1/\sigma, 1), V_1 \sim \chi^2_{n-1}, W_1 \text{ and } V_1 \text{ are independent}$

If the alternative $\theta_1 \neq 0$, $t(X_1, \ldots, X_n)$ has the non-central $t_{n-1}(\delta_1)$ with

$$\delta_1 = \sqrt{n}\theta_1/\sigma$$
Recall that we have $X_1, \ldots, X_n \sim N(\theta_1, \theta_3)$ and $Y_1, \ldots, Y_m \sim N(\theta_2, \theta_3)$

$H_0: \theta_1 = \theta_2$ vs. $H_1: \theta_1 \neq \theta_2$

The $T$-statistic mentioned in this example can be written as

$$T = \frac{W_2}{\sqrt{V_2/(n + m - 2)}}$$
$t$-test for comparison of two means

Here,

$$W_2 = \sqrt{\frac{nm}{n + m}}(\bar{X} - \bar{Y})/\sigma$$

$$V_2 = \sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2$$

Clearly, $W_2 \sim N\left(\sqrt{nm/(n + m)}(\theta_1 - \theta_2, 1), V_2 \sim \chi^2_{n+m-2}\right)$ and they are independent...so

When $\theta_1 \neq \theta_2$, $T \sim t_{n+m-2}(\delta_2)$ and

$$\delta_2 = \sqrt{nm/(n + m - 2)}(\theta_1 - \theta_2)/\sigma$$
Robustness of t-tests to violation of normality

- One-sample t-test: \( X_1, \ldots, X_n \sim L(\theta_1, \sigma^2) \) with \( L \) some distribution law
- Test \( H_0 : \theta_1 = \theta_1' \) vs. \( H_1 : \theta_1 \neq \theta_1' \)
- The t-test statistic is

\[
T_n = \frac{\sqrt{n}(\bar{X} - \theta_1')}{S_n}
\]

and the critical region is \( C_1 = \{| T_n | \geq t_{\alpha/2, n-1} \} \)
Robustness of t-tests to violation of normality

- $S_n \to \sigma$ in probability; by CLT,

$$T_n = \frac{\sigma}{S_n} \frac{\sqrt{n}(\bar{X} - \theta'_1)}{\sigma} \stackrel{D}{\to} Z$$

with $Z \sim N(0, 1)$

- In practice we will use a more conservative critical region $C_1$ and not $C_2 = \{ T_n | \geq z_{\alpha/2} \}$

- The $t$-test possesses robustness of validity but may not possess robustness of power
Test for equality of variances

- $X_1, \ldots, X_n \sim N(\theta_1, \theta_3)$ and $Y_1, \ldots, Y_m \sim N(\theta_2, \theta_4)$
- $\Omega = \{(\theta_1, \theta_2, \theta_3, \theta_4) : -\infty < \theta_1, \theta_2 < \infty, 0 < \theta_3, \theta_4 < \infty\}$
- $H_0 : \theta_3 = \theta_4$ vs. a general alternative
- $\omega = \{(\theta_1, \theta_2, \theta_3, \theta_4) : -\infty < \theta_1, \theta_2 < \infty, 0 < \theta_3 = \theta_4 < \infty\}$
Test for equality of variances

- The likelihood ratio statistic is
  \[ \Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})} \]
  is a function of
  \[ F = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)}{\sum_{i=1}^{m}(Y_i - \bar{Y})^2/(m-1)} \]

- Under \( H_0 \), \( F \sim F_{n-1,m-1} \); reject \( H_0 \) if \( F \leq c_1 \) or \( F \geq c_2 \)
- Typically, if \( \theta_3 = \theta_4 \), we select \( c_1 \) and \( c_2 \) s.t.
  \[ P(F \leq c_1) = P(F \geq c_2) = \frac{\alpha_1}{2} \]