STAT 517:Sufficiency

Uniformly most powerful tests (UMP) and likelihood ratio tests

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Definition

- Typically, it is important to handle the case where the alternative hypothesis may be a **composite** one
- It is desirable to have the best critical region for testing H_0 against each simple hypothesis in H_1
- The critical region C is **uniformly most powerful (UMP)** of size α against H_1 if it is so against *each* simple hypothesis in H_1
- A test defined by such a regions is a **uniformly most powerful(UMP)** test.
- Such a test does not always exist

Example

- $X_1, \ldots, X_n \sim N(0, \theta)$
- $H_0 : \theta = \theta^{'}$ vs. $H_1 : \theta > \theta^{'}$
- Note that the likelihood is

$$L(\theta; x_1, \ldots, x_n) = \left(\frac{1}{2\pi\theta}\right)^{n/2} \exp\left\{-\frac{1}{2\theta}\sum_{i=1}^{n} x_i^2\right\}$$

- Verify that the critical region $C : \{\mathbf{x} : \sum_{i=1}^{n} x_i^2 \geq c\}$ is the best critical region for such a test
- To find the critical value of c, recall that $\sum_{i=1}^{n} X_i^2/\theta^{'} \sim \chi_n^2$

Example

- If $X_1, \ldots, X_n \sim N(\theta, 1)$ and we test $H_0 : \theta = \theta^{'}$ vs. $H_1 : \theta \neq \theta^{'}$ there is no UMP
- Verify that there are two rejection regions:

$$\sum_{i=1}^{n} x_i \geq \frac{n}{2}(\theta^{''} - \theta^{'}) - \frac{\log k}{\theta'' - \theta'}$$

  if $\theta^{''} > \theta^{'}$ and
- $$\sum_{i=1}^{n} x_i \leq \frac{n}{2}(\theta^{''} - \theta^{'}) - \frac{\log k}{\theta'' - \theta'}$$

  if $\theta^{''} < \theta^{'}$
- If the alternative $H_1 : \theta > \theta^{'}$ or $H_1 : \theta < \theta^{'}$, the UMP does exist

Example: a UMP test for Poisson means

- Take $X_1, \ldots, X_n \sim P(\theta)$, $H_0 : \theta = 0.1$ vs. $H_1 : \theta > 0.1$
- The alternative $\theta'' > 0.1$ the likelihood ratio can be represented as

$$\left(\frac{0.1}{\theta''}\right)^{\sum x_i} e^{-10(1-\theta'')} \leq k$$

- The equivalent form is

$$\sum_{i=1}^{n} x_i \geq \frac{\log k + 10 - 10\theta''}{\log 0.1 - \log \theta''}$$

- The best critical region has the form $\sum_{i=1}^{n} x_i \geq c$ for a constant c

A useful remark

- If $X_1, \ldots, X_n \sim f(x;\theta)$, $Y = u(X_1, \ldots, X_n)$ is a sufficient statistic for θ, the ratio

$$\frac{L(\theta^{'}; x_1, \ldots, x_n)}{L(\theta^{''}; x_1, \ldots, x_n)} = \frac{k_1[u(x_1, \ldots, x_n); \theta^{'}]}{k_1[u(x_1, \ldots, x_n); \theta^{''}]}$$

  by factorization theorem
- If a best test or a UMP test is desired, only functions of sufficient statistics should be considered

General form of uniformly most powerful tests

- Take a general one-sided hypothesis: $H_0 : \theta \leq \theta^{'}$ vs. $H_1 : \theta > \theta^{'}$
- The level of the test is

$$\max_{\theta \leq \theta^{'}} \gamma(\theta)$$

the maximum probability of Type I error

- The general likelihood function is

$$L(\theta, \mathbf{x}) = \prod_{i=1}^{n} f(x_i; \theta)$$

General statement

- The likelihood $L(\theta;\mathbf{x})$ has the **monotone likelihood ratio** in statistic $y=u(\mathbf{x})$ for $\theta_1<\theta_2$ if

$$\frac{L(\theta_1;\mathbf{x})}{L(\theta_2;\mathbf{x})}$$

  is a monotone function of $y=u(\mathbf{x})$
- If our likelihood $L(\theta;\mathbf{x})$ has a monotone decreasing likelihood ratio in $y=u(\mathbf{x})$, the test "Reject H_0 if $Y\geq c_Y$" is the UMP
- Here, $\alpha=P_{\theta'}[Y\geq c_Y]$

Example: Bernoulli UMP

- Take $X_1, \ldots, X_n \sim b(1, \theta)$, $0 < \theta < 1$.
- For $\theta^{'} < \theta^{''}$, the likelihood ratio is

$$\left[\frac{\theta^{'}(1-\theta^{''})}{\theta^{''}(1-\theta^{'})}\right]^{\sum x_i}\left(\frac{1-\theta^{'}}{1-\theta^{''}}\right)^{n}$$

- $\frac{\theta^{'}(1-\theta^{''})}{\theta^{''}(1-\theta^{'})} < 1$ and so the ratio is a decreasing function of $y = \sum x_i$
- We have the MLR in $Y = \sum X_i$

Example: Bernoulli UMP

- The conclusion: the UMP level α decision rule to test H_0 vs. H_1 is: reject H_0 when

$$Y = \sum_{i=1}^{n} X_i \geq c$$

- Here, c is s. t. $\alpha = P_{\theta'}(Y \geq c)$

General UMP for regular exponential families

- The regular family pdf is

$$f(x;\theta) = \exp[p(\theta)K(x) + H(x) + q(\theta)]$$

- The likelihood ratio is

$$\frac{L(\theta^{'}}{L(\theta^{''})} = \exp\left\{[p(\theta^{'}) - p(\theta^{''})]\sum_{i=1}^{n} K(x_i) + n[q(\theta^{'}) - q(\theta^{''})]\right\}$$

- If $p(\theta)$ is an increasing function of θ, the ratio is a decreasing function of $y = \sum_{i=1}^{n} K(x_i)$
- For $H_0 : \theta \leq \theta^{''}$ vs. $H_1 : \theta > \theta^{'}$, the UMP level α decision rule is to reject H_0 if

$$Y = \sum_{i=1}^{n} K(X_i) \geq c$$

- Here c is s.t. $\alpha = P_{\theta^{'}}(Y \geq c)$

The same for an opposite setup

- Now, let $H_0 : \theta \geq \theta^{'}$ vs. $H_1 : \theta < \theta^{'}$
- The UMP level α decision rule is (again, if $p(\theta)$ is an increasing function) to reject H_0 if

$$Y = \sum_{i=1}^{n} K(X_i) \leq c$$

- Here, c is s. t. $\alpha = P_{\theta'}(Y \leq c)$

The case of a simple null hypothesis

- If $H_0 : \theta = \theta^{'}$ and $H_1 : \theta > \theta^{'}$, $\sum_{i=1}^{n} K(x_i) \geq c$ is a UMP critical region
- All of the regions considered earlier: $\sum_{i=1}^{n} x_i^2 \geq c$, $\sum_{i=1}^{n} x_i \geq c$ etc. are UMP for testing H_0 vs H_1

General Likelihood Ratio Tests: a possible setting

- What if $H_0 : \boldsymbol{\theta} \in \omega$ and $H_1 : \boldsymbol{\theta} \in \Omega \cap \omega^{'}$?
- Recall testing the normal mean with unknown variance
- $X_1, \dots, X_n \sim N(\theta_1, \theta_2)$, $\boldsymbol{\theta} = (\theta_1, \theta_2)$
- $\Omega = \{\boldsymbol{\theta} : -\infty < \theta_1 < \infty, \theta_2 > 0\}$, $\omega = \{\boldsymbol{\theta} : \theta_1 = \theta_1^{'}, \theta_2 > 0\}$

Possibilities

- If θ_2 is known, we found earlier that there is no UMP test
- It is possible to construct a theory of best tests if unbiasedness is required
- A specific critical region will be

$$C_2 = \left\{ |\bar{X} - \theta_1^{'}| > \sqrt{\frac{\theta_2}{n}} z_{\alpha/2} \right\}$$

- C_2 is unbiased and provides a UMP unbiased test of level α

Comparison of two normal means

- $X_1, \dots, X_n \sim N(\theta_1, \theta_3)$, $Y_1, \dots, Y_n \sim N(\theta_2, \theta_3)$
- All of the parameters are unknown
- $H_0 : \theta_1 = \theta_2$
- Then, $\Omega = \{(\theta_1, \theta_2, \theta_3) : -\infty < \theta_1, \theta_2 < \infty, \theta_3 > 0\}$
- $\omega = \{(\theta_1, \theta_2, \theta_3) : \theta_1 = \theta_2, \theta_3 > 0\}$

MLE

- $$L(\omega) = \left(\frac{1}{2\pi\theta_3}\right)^{(n+m)/2} \times$$
$$\exp\left\{-\frac{1}{2\theta_3}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2+\sum_{i=1}^{m}(y_i-\theta_1)^2\right]\right\}$$

$$L(\Omega) = \left(\frac{1}{2\pi\theta_3}\right)^{(n+m)/2} \times$$
$$\exp\left\{-\frac{1}{2\theta_3}\left[\sum_{i=1}^{n}(x_i-\theta_1)^2+\sum_{i=1}^{m}(y_i-\theta_2)^2\right]\right\}$$

MLE

- Let ω and $\omega^{'}$ be the MLE solutions for the variance parameter θ_3
- The likelihood ratio is

$$\Lambda = \left(\frac{\omega^{'}}{\omega}\right)^{(n+m)/2}$$

- We can transform the above to find that

$$\Lambda^{2/(n+m)} = \frac{n+m-2}{(n+m-2)+T^2}$$

where $T \sim t_{n+m-2}$ under the null hypothesis
- The significance level of this test is

$$\alpha = P_{H_0}[\Lambda(X_1,\ldots,X_n,Y_1,\ldots,Y_n) \le \lambda_0]$$

- The above inequality is equivalent to $|T| \ge c$ so

$$\alpha = P(|T| \ge c; H_0)$$

Non-central t distribution

- Let $W \sim N(\delta, 1)$, $V \sim \chi^2_r$, W and V are independent
- The quotient
$$T = \frac{W}{\sqrt{V/r}}$$
has a **non-central** t distribution with r df and **noncentrality parameter** δ
- The usual t distribution is a special case with $\delta = 0$

t-test for the mean and its power

- Let $X_1, \ldots, X_n \sim N(\theta, \sigma^2)$; test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta \neq \theta_0$
- Recall

$$t(X_1, \ldots, X_n) = \frac{\sqrt{n}\bar{X}}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)}}$$

- Clearly,

$$t(X_1, \ldots, X_n) = \frac{\sqrt{n}\bar{X}/\sigma}{\sqrt{\sum_{i=1}^{n}(X_i - \bar{X})^2/[\sigma^2(n-1)]}} = \frac{W_1}{\sqrt{V_1/(n-1)}}$$

with $W_1 = \sqrt{n}\bar{X}/\sigma$, $V_1 = \sum_{i=1}^{n}(X_i - \bar{X})^2/\sigma^2$

t-test for the mean and its power

- $W_1 \sim N(\sqrt{n}\theta_1/\sigma, 1)$, $V_1 \sim \chi^2_{n-1}$, W_1 and V_1 are independent
- If the alternative $\theta_1 \neq 0$, $t(X_1, \ldots, X_n)$ has the non-central $t_{n-1}(\delta_1)$ with

$$\delta_1 = \sqrt{n}\theta_1/\sigma$$

t-test for comparison of two means

- Recall that we have $X_1, \ldots, X_n \sim N(\theta_1, \theta_3)$ and $Y_1, \ldots, Y_m \sim N(\theta_2, \theta_3)$
- $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$
- The T-statistic mentioned in this example can be written as

$$T = \frac{W_2}{\sqrt{V_2/(n+m-2)}}$$

t-test for comparison of two means

- Here,

$$W_2 = \sqrt{\frac{nm}{n+m}}(\bar{X} - \bar{Y})/\sigma$$

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$$V_2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{m}(Y_i - \bar{Y})^2}{\sigma^2}$$

- Clearly, $W_2 \sim N\left(\sqrt{nm/(n+m)}(\theta_1 - \theta_2, 1\right)$, $V_2 \sim \chi^2_{n+m-2}$ and they are independent...so
- When $\theta_1 \neq \theta_2$, $T \sim t_{n+m-2}(\delta_2)$ and

$$\delta_2 = \sqrt{nm/(n+m-2)}(\theta_1 - \theta_2)/\sigma$$

Robustness of t-tests to violation of normality

- One-sample *t*-test: $X_1, \ldots, X_n \sim L(\theta_1, \sigma^2)$ with L some distribution *law*
- Test $H_0 : \theta_1 = \theta_1^{'}$ vs. $H_1 : \theta_1 \neq \theta_1^{'}$
- The *t*-test statistic is

$$T_n = \frac{\sqrt{n}(\bar{X} - \theta_1^{'})}{S_n}$$

and the critical region is $C_1 = \{|T_n| \geq t_{\alpha/2,n-1}\}$

Robustness of t-tests to violation of normality

- $S_n \to \sigma$ in probability; by CLT,

$$T_n = \frac{\sigma}{S_n}\frac{\sqrt{n}(\bar{X} - \theta_1')}{\sigma} \xrightarrow{D} Z$$

  with $Z \sim N(0,1)$
- In practice we will use a more conservative critical region C_1 and not $C_2 = \{T_n| \geq z_{\alpha/2}\}$
- The *t*-test possesses **robustness of validity** but may not possess **robustness of power**

Test for equality of variances

- $X_1, \ldots, X_n \sim N(\theta_1, \theta_3)$ and $Y_1, \ldots, Y_m \sim N(\theta_2, \theta_4)$
- $\Omega = \{(\theta_1, \theta_2, \theta_3, \theta_4) : -\infty < \theta_1, \theta_2 < \infty, 0 < \theta_3, \theta_4 < \infty\}$
- $H_0 : \theta_3 = \theta_4$ vs. a general alternative
- $\omega = \{(\theta_1, \theta_2, \theta_3, \theta_4) : -\infty < \theta_1, \theta_2 < \infty, 0 < \theta_3 = \theta_4 < \infty\}$

Test for equality of variances

- The likelihood ratio statistic is

$$\Lambda = \frac{L(\hat{\omega})}{L(\hat{\Omega})}$$

  is a function of

$$F = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2/(n-1)}{\sum_{i=1}^{m}(Y_i - \bar{Y})^2/(m-1)}$$

- Under H_0, $F \sim F_{n-1,m-1}$; reject H_0 if $F \leq c_1$ or $F \geq c_2$
- Typically, if $\theta_3 = \theta_4$, we select c_1 and c_2 s.t.

$$P(F \leq c_1) = P(F \geq c_2) = \frac{\alpha_1}{2}$$