STAT 517: Optimal tests of hypotheses

Most powerful tests

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Setting

- $X \sim f(x;\theta)$ where $\theta \in \Omega$
- For two disjoint ω_0 and ω_1 $\Omega = \omega_0 \cup \omega_1$
- Hypotheses: $H_0 : \theta \in \omega_0$ vs. $H_1 : \theta \in \omega_1$
- H_0 is the **null hypothesis** and H_1 is the **alternative hypothesis**
- The test is based on a sample $\mathbf{X} = (X_1, \ldots, X_n)'$ from the distribution of X; $\mathbf{x} = (x_1, \ldots, x_n)'$ are the values of this sample

Setting

- A **test** of H_0 vs. H_1 is based on the subset of the support of $\mathbf{X}'$ $C \subset \mathcal{S}$
- The set C is the **critical region** if the decision rule is
  - Reject H_0 if $\mathbf{X} \in C$
  - Don't reject H_0 if $\mathbf{X} \in C'$

Types of errors

- A **Type I** error occurs if H_0 is rejected while it is true
- A **Type II** error occurs of H_0 isn't rejected while an alternative is true
- The **size** or **significance level** of the test is the probability of Type I error:

$$\alpha = \max_{\theta \in \omega_0} P_\theta(X \in C)$$

- The **power function** of the test is

$$\gamma_C(\theta) = P_\theta(X \in C)$$

- Subject to test having size α select tests that minimize the probability of Type II error

The best critical region

- When testing $H_0 : \theta = \theta^{'}$ vs. $H_1 : \theta = \theta^{''}$ the **best critical region** of size α is defined as C s.t.
  - $P_{\theta'}(X \in C) = \alpha$
  - For every subset A of the sample space

$$P_{\theta'}(X \in A) \to P_{\theta''}(X \in C) \geq P_{\theta''}(X \in A)$$

Neyman-Pearson theorem

- $X_1, \ldots, X_n \sim f(x;\theta)$ and the likelihood ois

$$L(\theta;\mathbf{x}) = \prod_{i=1}^{n} f(x_i;\theta)$$

- The parameter space $\Omega = \{\theta', \theta''\}$, some $k > 0$
- C is a subset of the sample space s.t.
  - $\frac{L(\theta';\mathbf{x})}{L(\theta'';\mathbf{x})} \le k$ for each $\mathbf{x} \in C$
  - $\frac{L(\theta';\mathbf{x})}{L(\theta'';\mathbf{x})} \ge k$ for each $\mathbf{x} \in C'$
  - $\alpha = P_{H_0}(X \in C)$
- Then, C is the best critical region of size α

Unbiased tests

- A test with the critical region C and level α is called **unbiased** if
$$P_\theta(X \in C) \geq \alpha$$
for all $\theta \in \omega_1$
- C is the critical region of the best test of $H_0 : \theta = \theta^{'}$ vs. $H_1 : \theta = \theta^{''}$
- $\gamma_C(\theta^{''})$ is the power of the test
- Then $\alpha \leq \gamma_C(\theta^{''})$

Remark

- Note that inequality $\frac{L(\theta^{'};\mathbf{x})}{L(\theta^{''};\mathbf{x})} \le k$ can often be expressed as

$$u_1(\mathbf{x};\theta^{'},\theta^{''}) \le c_1$$

or

$$u_2(\mathbf{x};\theta^{'},\theta^{''}) \ge c_2$$

- If the distribution of this new statistic can be found, we determine c_1 from

$$\alpha = P_{H_0}(u_1(\mathbf{X};\theta^{'},\theta^{''}) \le c_1)$$