# STAT 517:Sufficiency

## Exponential families of distributions. Estimation of functions of parameters

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April 16, 2015

# Definition

- Let the pdf $f(x;\theta)$ have the form

$$f(x;\theta) = \exp[p(\theta)K(x) + H(x) + q(\theta)$$

  when $x \in S$ and zero otherwise ($S$ is the support of $X$)
- Such a pdf is said to be a member of the **regular exponential family (class)** if
  - $S$ does not depend upon $\theta$
  - $p(\theta)$ is a non-trivial continuous function of $\theta \in \Omega$ called a **natural parameter**
    - Each of $K^{'}(x) \neq 0$ and $H(x)$ is a continuous function of $x \in S$ if $X$ is a continuous random variable
    - $K(x)$ is a non-trivial function of $x \in S$ if $X$ is a discrete random variable

# Examples

- Let $f(x;\theta) = \theta e^{-\theta x}$, $x \geq 0$
- The support is $[0, \infty)$ for all $\theta$; $f(x;\theta)$ can be presented in the required form with $p(\theta) = \theta$, $K(x) = -x$, $q(\theta) = \log\theta$, $H(x) \equiv 0$
- Let $X \sim b(n,p)$, $0 < p < 1$ where $n$ is known; the support is $S = \{0, 1, \dots, n\}$ for all $p$.
- Moreover,

$$
\begin{aligned}
f(x;p) &= \binom{n}{x} p^x (1-p)^{n-x} \\
&= \exp\left\{ x \log\left(\frac{p}{1-p}\right) + n\log(1-p) + \log\binom{n}{x} \right\}
\end{aligned}
$$

# Examples

- Let $f(x;\theta)$ be $N(0,\theta)$ with $0 < \theta < \infty$
- Then,

$$f(x;\theta) = \frac{1}{\sqrt{2\pi\theta}} e^{-x^2/2\theta} = \exp\left(-\frac{1}{2\theta^2}x^2 - \log\sqrt{2\pi\theta}\right)$$

  and so $N(0,\theta)$ is a member of an exponential family
- On the other hand, let $f(x;\theta) = \frac{1}{\theta} = \exp\{-\log\theta\}$ when $0 < x < \theta$ and 0 elsewhere
- This uniform distribution does not belong to an exponential family since its support depends on $\theta$

# Exponential families and sufficient statistics

- The likelihood for $X_1, \ldots, X_n \sim f(x;\theta)$ is

$$\exp\left[p(\theta)\sum_{i=1}^{n} K(x_i) + \sum_{i=1}^{n} H(x_i) + nq(\theta)\right]$$

  and so $Y_1 = \sum_{i=1}^{n} K(X_i)$ is a sufficient statistic
- Thus, for any exponential family, a sufficient statistic can be simply identified by visual inspection!
- The pdf(pmf) of $Y_1$ is

$$f_{Y_1}(y_1;\theta) = R(y_1)\exp[p(\theta)y_1 + nq(\theta)]$$

  for $y_1 \in \mathcal{S}_{Y_1}$ and some function $R(y_1)$ neither of which depend on $\theta$

# Exponential families and sufficient statistics

- For any regular exponential family,

$$\mathbb{E}(Y_1) = -n\frac{q^{'}(\theta)}{p^{'}(\theta)}$$

and

$$Var(Y_1) = n\frac{1}{p^{'}(\theta)}\left\{p^{''}(\theta)q^{'}(\theta) - q^{''}(\theta)p^{'}(\theta)\right\}$$

- Thus, expectations and variances can also be found immediately from the general form of an exponential family

# Example

- Let $X \sim P(\theta)$ where $\theta \in (0, \infty)$and $\mathcal{S} = \{0, 1, 2, \dots\}$
- Write

$$f(x;\theta) = e^{-\theta}\frac{\theta^x}{x!} = \exp\{(\log\theta)x + \log(1/x!) + (-\theta)\}$$

  and so $p(\theta) = \log(\theta)$, $q(\theta) = -\theta$ and $K(x) = x$
- Clearly, $Y_1 = \sum_{i=1}^n X_i$ is a sufficient statistic with the mean equal to $-n\frac{q^{'}(\theta)}{p^{'}(\theta)} = n\theta$ and the same variance
- Also easy to show that $R(y_1) = n^{y_1}(1/y_1!)$

# Exponential families and complete sufficient statistics

- If $f(x;\theta)$, for $\gamma < \theta < \delta$ belongs to a regular exponential family, $X_1, \dots, X_n \sim f(x;\theta)$ , $Y_1 = \sum_{i=1}^n K(X_i)$ is a complete sufficient statistic for $\theta$
- As an example, consider $X_1, \dots, X_n \sim N(\theta, \sigma^2)$ with known $\sigma^2 > 0$
- Verify that $p(\theta) = \frac{\theta}{\sigma^2}$, $K(x) = x$ etc and so $Y_1 = \sum_{i=1}^n X_i$ is a complete sufficient statistic for $\theta$
- Also, $\phi(Y_1) = \frac{Y_1}{n} = \bar{X}$ is an unbiased estimator of $\theta$ and, thus, is a unique MVUE
- Note that $\bar{X}$ itself is also a complete sufficient statistic
- By the same reasoning, we can find that for Poisson distribution $\bar{X}$ is also a unique MVUE for the parameter $\theta$ as well

# Estimating functions of parameters using the expected value of sufficient statistics

- Let $X_1, \ldots, X_n \sim b(1, \theta)$ for $0 < \theta < 1$
- We know that $\bar{X}$ is the unique MVUE of $\theta$ and $Var\, \bar{X} = \frac{\theta(1-\theta)}{n} = \frac{\delta}{n}$. How do we estimate this variance?
- $\bar{X}$ is a sufficient statistic for $\theta$, we can limit the choice to functions of $\bar{X}$; the MLE of $\delta$ is $\tilde{\delta} = \bar{X}(1 - \bar{X})$
- To check if $\frac{\tilde{\delta}}{n}$ is an MVUE of $Var\, \bar{X}$ we need to find its expectation first

# Estimating functions of parameters using the expected value of sufficient statistics

- Check that
$$\mathbb{E}[\tilde{\delta}] = (n-1)\frac{\theta(1-\theta)}{n}$$
and so $\hat{\delta} = \frac{n}{n-1}(\bar{X})(1-\bar{X})$ is the unique MVUE of $\delta$
- We can also say that the unique MVUE of $\delta/n$ is $\tilde{\delta}/n$

# Estimating functions of parameters using the conditional expectation of an unbiased estimate w.r.t a sufficient statistic

- The problem: estimating quantiles of $N(\theta, 1)$ given $X_1, \ldots, X_n$ with $n > 1$
- Mathematically, need to estimate $P(X \leq c)$ for some fixed (and known $c$)
- A sensible unbiased estimator is a function $u(X_1) = 1$ if $X_1 \leq c$ and 0 if $X_1 > c$

# Estimating functions of parameters using the conditional expectation of an unbiased estimate w.r.t a sufficient statistic

- The joint distribution of $X_1$ and $\bar{X}$ is, of course, bivariate normal with the mean vector $(\theta, \theta)$, variances $\sigma_1^2 = 1$, $\sigma_2^2 = \frac{1}{n}$ and $\rho = \frac{1}{\sqrt{n}}$
- The conditional distribution of $X_1$ given $\bar{X}$ is normal with linear conditional mean
$$\theta + \rho \frac{\sigma_1}{\sigma_2}(\bar{x} - \theta) = \bar{x}$$
and the variance
$$\sigma_1^2(1 - \rho^2) = \frac{n-1}{n}$$
- The direct calculation gives the conditional expectation as
$$\phi(\bar{x}) = \Phi\left[\frac{\sqrt{n}(c - \bar{x})}{\sqrt{n-1}}\right]$$

# MVUE principle

- The MVUE principle is not a theorem and there are cases where its application results in an unsatisfactory results
- E.g. take $X \sim P(\theta)$ , $0 < \theta < \infty$; thus, $X$ is a sample size 1 from this Poisson distribution
- $X$ is a complete sufficient statistic; if we want to obtain an MVUE of $e^{-2\theta}$, consider a candidate $Y = 1^{-X}$
- $$\mathbb{E}\, Y = \sum_{x=0}^{\infty} \frac{(-\theta)^x e^{-\theta}}{x!} = e^{-2\theta}$$

  and so $Y$ is an MVUE
- Nevertheless, it can only take either 1 or $-1$ as its values...hardly a good estimator
- Note that the MLE in this case is $e^{-2X}$ which looks much better!